Testing Linear Hypothesis about β; Prediction

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Part I

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Literature:

Wooldridge, Chapter 4
Johnston & DiNardo, 1.5.3, 1.6, 1.7, 3.4, 3.5

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1 Testing Linear Hypothesis about β

There are a number of different tests on β that may be relevant:

1. Test significance of a regressor x_k's influence on y, i.e., test if the regressor has an impact on the dependent variable. This is what is most commonly referred to as a *significance test.*

$$H_0 : \beta_k = 0$$

2. Test that β_k is equal to some specific value:

$$H_0 : \beta_k = \beta_{k0}$$

3. Test significance of the regression:

$$\begin{bmatrix} \beta_2 \\ \beta_3 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

4. Test that a subset of vectors has no influence on the explanatory variable:

$$H_0 : \boldsymbol{\beta_2} = 0$$

where the $\boldsymbol{\beta}$ vector is partioned into two subvectors, $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ containing respectively k_1 and $k_2 = k - k_1$ elements.

5. Others...

$$\begin{aligned} H_0 &: \beta_2 + \beta_3 = 1 \\ H_0 &: \beta_3 = \beta_4 \end{aligned}$$

All of the aove tests fall into a general linear framework:

$$\boldsymbol{R\beta} = \boldsymbol{r}$$

Need to test the general linear hypothesis:

$$H_0 : \boldsymbol{R\beta} - \boldsymbol{r} = \boldsymbol{0}$$

The vector $(\boldsymbol{R\beta} - \boldsymbol{r})$ measures the discrepancy between expectation and observation. If this discrepancy is large, doubt is cast on the null hypothesis. On the other hand, if the discrepancy is small, it tends not to contradict the null. What is "large" and "small" is determined from the relevant sampling distribution, in this case the distribution of $\boldsymbol{Rb}$ when $\boldsymbol{R\beta} = \boldsymbol{r}$.

$$\begin{aligned} E[\boldsymbol{Rb}] &= \boldsymbol{R\beta} \\ var(\boldsymbol{Rb}) &= E[\boldsymbol{R}(\boldsymbol{b} - \boldsymbol{\beta})(\boldsymbol{b} - \boldsymbol{\beta})'\boldsymbol{R}'] \\ &= \boldsymbol{R}var(\boldsymbol{b})\boldsymbol{R}' \\ &= \sigma^2\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}' \end{aligned}$$

Since $\boldsymbol{b}$ is a function of ϵ and

$$\epsilon \sim N(\boldsymbol{0}, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1})$$

it follows that:

$$\begin{aligned} \boldsymbol{b} &\sim N[\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}] \\ \boldsymbol{Rb} &\sim N[\boldsymbol{R\beta}, \sigma^2\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}'] \\ \boldsymbol{R}(\boldsymbol{b} - \boldsymbol{\beta}) &\sim N[\boldsymbol{0}, \sigma^2\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}'] \end{aligned}$$

If the null hypothesis is true: $\boldsymbol{R\beta} = \boldsymbol{r}$ then:

$$(\boldsymbol{Rb} - \boldsymbol{r}) \sim N[\boldsymbol{0}, \sigma^2\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}']$$

from which we can derive a χ^2 variable:

$$(\boldsymbol{Rb} - \boldsymbol{r})'[\sigma^2\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}']^{-1}(\boldsymbol{Rb} - \boldsymbol{r}) \sim \chi^2(q)$$

The only problem with deriving the χ^2 statistic above is the presence of the unkown σ^2. However it can be shown that:

$$\frac{\boldsymbol{e}'\boldsymbol{e}}{\sigma^2} \sim \chi^2(n - k)$$

Combining these equation:

$$\frac{(\boldsymbol{Rb} - \boldsymbol{r})'[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}']^{-1}(\boldsymbol{Rb} - \boldsymbol{r})/q}{\frac{\boldsymbol{e}'\boldsymbol{e}}{(n-k)}}$$

or:

$$(\boldsymbol{Rb}-\boldsymbol{r})'[s^2\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}']^{-1}(\boldsymbol{Rb}-\boldsymbol{r})/q$$

The test procedure is then to reject the null hypothesis ($\boldsymbol{R\beta}=\boldsymbol{r}$) if the computed F value exceeds a predetermined critical vaule.

Example:

$$H_0:\beta_2=\beta_3=\cdots\beta_k=0$$

This is a test of a composite hypothesis about $k-1$ regressor coefficients. Partion the $\boldsymbol{X}$ matrix into a column vector of ones ($\boldsymbol{i}$) and a matrix of the k explanatory variables $\boldsymbol{X_2}$ such that:

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix}\boldsymbol{i} & \boldsymbol{X_2}\end{bmatrix}'\begin{bmatrix}\boldsymbol{i} & \boldsymbol{X_2}\end{bmatrix} = \begin{bmatrix} n & \boldsymbol{i}'\boldsymbol{X_2} \\ \boldsymbol{X_2}'\boldsymbol{i} & \boldsymbol{X_2}'\boldsymbol{X_2}\end{bmatrix}$$

Now $\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}'$ picks out the submatrix of order $k-1$ in the bottom corner of $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ where $\boldsymbol{X_2}$ is the matrix of observations on all $k-1$ regressors.
Note that the submatrix we are interested in can be expressed as:

$$[\boldsymbol{X_2}'\boldsymbol{X_2}-\boldsymbol{X_2}'\boldsymbol{i}n^{-1}\boldsymbol{i}'\boldsymbol{X_2}]^{-1}=[\boldsymbol{X_2}'\boldsymbol{A}\boldsymbol{X_2}]^{-1}=[\boldsymbol{X_*}'\boldsymbol{X_*}]^{-1}$$

Where $\boldsymbol{A}$ is the transformation matrix and $\boldsymbol{X_*}=\boldsymbol{A}\boldsymbol{X_2}$.

As $\boldsymbol{Rb}=\boldsymbol{b_2}$ and $\boldsymbol{r}=\boldsymbol{0}$:

$$F=\frac{\boldsymbol{b_2}'\boldsymbol{X_*}'\boldsymbol{X_*}\boldsymbol{b_2}/(k-1)}{\boldsymbol{e}'\boldsymbol{e}/(n-k)}\sim F(k-1,n-k)$$

which can also be expressed as:

$$F=\frac{ESS/(k-1)}{RSS/(n-k)}\sim F(k-1,n-k)$$

or:

$$F=\frac{R^2/(k-1)}{(1-R^2)/(n-k)}\sim F(k-1,n-k)$$

The above example is a special case of the more general case where subsets of coefficients are tested. Partition the regression equation as follows:

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{X_1} & \boldsymbol{X_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \end{bmatrix} + \boldsymbol{e} = \boldsymbol{X_1 b_1} + \boldsymbol{X_2 b_2} + \boldsymbol{e}$$

where $\boldsymbol{X_1}$ has k_1 columns including a column of ones and $\boldsymbol{X_2}$ has $k_2 = k - k_1$ columns. $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ are corresponding vectors of regression coefficients.

When testing $H_0 : \boldsymbol{b_2} = \boldsymbol{0}$, another method is to run two separate regressions:

1. First regress $\boldsymbol{y}$ on $\boldsymbol{X_1}$ and denote the residual som of squares as $\boldsymbol{e_*}'\boldsymbol{e_*}$. This is the so-called *restricted* regression and $\boldsymbol{e_*}'\boldsymbol{e_*}$ the *restricted* RSS.

2. Then regress $\boldsymbol{y}$ on all regressors ($\boldsymbol{X_1}$ and $\boldsymbol{X_2}$). This is the *unrestricted* regression as no restrictions are placed on the $\boldsymbol{b}$ vector. The test statistic is then:

$$F = \frac{\boldsymbol{e_*}'\boldsymbol{e_*} - \boldsymbol{e}'\boldsymbol{e}/(k_2)}{\boldsymbol{e}'\boldsymbol{e}/(n-k)} \sim F(k_2, n-k)$$

2 Prediction

Suppose that a regression equation has been fitted and one wants to use this equation to predict $\boldsymbol{y}$ with some specific regressor values:

$$\boldsymbol{c}' = [1 X_{2f} X_{3f} \cdots X_{kf}]$$

A point prediction is obtained by inserting the given X values into the regression equation:

$$\hat{Y}_f = b_1 + b_2 X_{2f} + b_3 X_{3f} + \cdots + b_k X_{kf} = \boldsymbol{c}'\boldsymbol{b}$$

where $\boldsymbol{c}'\boldsymbol{b}$ is a best linear unbiased estimator of $\boldsymbol{c}'\boldsymbol{\beta} = E[Y_f]$.

$$var(\boldsymbol{c}'\boldsymbol{b}) = \boldsymbol{c}' var(\boldsymbol{b})\boldsymbol{c}$$

Assuming normality for the disturbance term, it follows that:

$$\frac{\boldsymbol{c}'\boldsymbol{b} - \boldsymbol{c}'\boldsymbol{\beta}}{\sqrt{var(\boldsymbol{c}'\boldsymbol{b})}}$$

Replacing the unkown σ^2 in the above equation by s^2:

$$\frac{\hat{Y}_f - E(Y_f)}{s\sqrt{\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{c}}} \sim t(n-k)$$

from which a 95 procent confidence interval can be calculated for $E(Y_f)$:

$$\hat{Y}_f \quad \pm \quad t_{0.025}s\sqrt{\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{c}}$$