Local and Consistent Centrality Measures in Networks[1]

Vianney Dequiedt[2] and Yves Zenou[3]

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Abstract

The centrality of an agent in a network has been shown to be crucial in explaining different behaviors and outcomes. In this paper, we propose an axiomatic approach to characterize a class of centrality measures for which the centrality of an agent is recursively related to the centralities of the agents she is connected to. This includes the Katz-Bonacich and the eigenvector centrality. The core of our argument hinges on the power of the consistency axiom, which relates the properties of the measure for a given network to its properties for a reduced problem. In our case, the reduced problem only keeps track of local and parsimonious information. Our axiomatic characterization highlights the conceptual similarities among this class of measures.

Keywords: Consistency, centrality measures, networks, axiomatic approach.

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[2]CERDI, Université d'Auvergne, France. E-mail: vianney.dequiedt@udamail.fr.

[3]Stockholm University and IFN. E-mail: yves.zenou@ne.su.se.

1 Introduction

Centrality is a fundamental concept in network analysis. Bavelas (1948) and Leavitt (1951) were among the first to use centrality to explain differential performance of communication networks and network members on a host of variables including time to problem solution, number of errors, perception of leadership, efficiency, and job satisfaction.

Following their work, many researchers have investigated the importance of the centrality of agents on different outcomes. Indeed, it has been shown that centrality is important in explaining employment opportunities (Granovetter, 1974), exchange networks (Cook et al., 1983; Marsden, 1982), peer effects in education and crime (Calvó-Armengol et al., 2009; Haynie, 2001), power in organizations (Brass, 1984), the adoption of innovation (Coleman et al., 1966), the creativity of workers (Perry-Smith and Shalley, 2003), the diffusion of microfinance programs (Banerjee et al., 2013), the flow of information (Borgatti, 2005; Stephenson and Zelen, 1989), the formation and performance of R&D collaborating firms and inter-organizational networks (Boje and Whetten, 1981; Powell et al., 1996; Uzzi, 1997), the success of open-source projects (Grewal et al., 2006) as well as workers' performance (Mehra et al., 2001).

While many measures of centrality have been proposed,[4] the category itself is not well defined beyond general descriptors such as node prominence or structural importance. There is a class of centrality measures, call prestige measures of centrality, where the centralities or statuses of positions are recursively related to the centralities or statuses of the positions to which they are connected. Being chosen by a popular individual should add more to one's popularity. Being nominated as powerful by someone seen by others as powerful should contribute more to one's perceived power. Having power over someone who in turn has power over others makes one more powerful. This is the type of centrality measure that will be the focus of this paper.

This class of centrality measures includes the degree centrality, the Katz-Bonacich centrality (due to Katz, 1953, and Bonacich, 1987) and the eigenvector centrality. Take, for example, the Katz-Bonacich centrality of a particular node. It counts the total number of walks that start from this node in the graph, weighted by a decay factor based on path length. This means that the walks are weighted inversely by

[4]See Wasserman and Faust (1994) and Jackson (2008) for an introduction and survey.

their length so that long, highly indirect walks count for little, while short, direct walks count for a great deal. Another way of interpreting this walk-based measure is in terms of an intuitive notion that a persons centrality should be a function of the centrality of the people he or she is associated with. In other words, rather than measure the extent to which a given actor "knows everybody", we should measure the extent to which the actor "knows everybody who is anybody".

While there is a very large literature in mathematical sociology on centrality measures (see e.g. Borgatti and Everett, 2006; Bonacich and Loyd, 2001; Wasserman and Faust, 1994), little is known about the foundation of this class of centrality measures from a behavioral viewpoint.[5] Ballester et al. (2006) were the first to provide a microfoundation for the Katz-Bonacich centrality. They show that, if the utility of each agent is linear-quadratic, then, under some condition, the unique Nash equilibrium in pure strategies of a game where n agents embedded in a network simultaneously choose their effort level is such that the equilibrium effort is equal to the Katz-Bonacich centrality of each agent. This result is true for any possible connected network of n agents.[6] In other words, Nash is Katz-Bonacich and the position of each agent in a network fully explains her behavior in terms of effort level.

In the present paper, we investigate further the importance of centrality measures in economics by adopting an *axiomatic* approach. We derive characterization results not only for the Katz-Bonacich centrality but also for other centrality measures that have the properties that one's centrality can be deduced from one's set of neighbors and their centralities. This class includes the degree centrality and the eigenvector centrality.

Our characterization results are based on three key ingredients, namely the definitions of an *embedded* network and of a *reduced embedded* network and the *consistency* property.

An embedded network is defined as a set of nodes and links for which some of the nodes, that we call *terminal* nodes, are assigned a positive real number. We further require that the set of terminal nodes forms an independent set, i.e. that no two terminal nodes are linked and that each terminal node is the neighbor of at least one *regular*, i.e. a non-terminal node. Conceptually, one can interpret an embedded

[5]For a survey of the literature on networks in economics, see Jackson (2008, 2014), Ioannides (2012), Jackson and Zenou (2014) and Jackson et al. (2014).

[6]With undirected links among n agents, there are $2^{n(n-1)/2}$ possible networks.

network as a set of regular nodes and their neighbors such that the centrality of some of those neighbors, the terminal nodes, has been parameterized and no longer needs to be determined.

A reduced embedded network is defined from an initial embedded network together with a vector of centralities. It is a small world that consists in a subset of regular nodes of the initial embedded network and their neighbors. The terminal nodes in the reduced network are assigned a positive number which is either kept from the initial network or taken from the vector of centralities.

Those two definitions are instrumental in order to characterize centrality measures when combined with the *consistency* property. This property requires that the centralities in the initial network are also the centralities in the reduced networks constructed from the initial network and its vector of centralities.

As stressed by Aumann (1987), consistency is a standard property in cooperative game as well as noncooperative game theory. It has been used to characterize the Nash equilibrium correspondence (Peleg and Tijs, 1996), the Nash bargaining solution (Lensberg, 1988), the core (Peleg, 1985) and the Shapley value (Hart and Mas-Colell, 1989; Maschler and Owen, 1989) to name a few. As nicely exposed by Thomson (2011), consistency expresses the following idea. A measure is consistent if for any network in the domain and the "solution" it proposes for this network, the "solution" for the reduced network obtained by envisioning the departure of a subset of regular nodes with their component of the solution is precisely the restriction of the initial solution to the subset of remaining regular nodes. Consistency can be seen as a robustness principle, it requires that the measure gives coherent attributes to nodes as the network varies.

The usefulness of the consistency property for characterization purposes depends on how a reduced problem is defined. In our case, it is very powerful since a reduced problem only keeps track of local and parsimonious information, namely the set of neighbors and the centrality of those neighbors.

Contrary to the Nash equilibrium approach (Ballester et al., 2006), we believe that our *axiomatic* approach allows us to understand the relationship between different centrality measures belonging to the same class, i.e. the degree, the Katz-Bonacich and the eigenvector centrality measure. This is important because as stated above, different types of centralities can explain different behaviors and outcomes. For example, the eigenvector centrality seems to be important in the diffusion of a microfinance program in India (Banerjee et al., 2013). On the contrary, the Katz-Bonacich centrality seems to be crucial in explaining educational and crime outcomes (Haynie, 2001; Calvó-Armengol et al., 2009) and, more generally, outcomes for which complementarity in efforts matter. The degree centrality is also important. For example, Christakis and Fowler (2010) combine Facebook data with observations of a flu contagion, showing that individuals with more friends were significantly more likely to be infected at an earlier time than less connected individuals.

The axiomatic approach is a standard approach in the cooperative games and social choice literature but axiomatic characterizations of centrality measures are scarce. Boldi and Vigna (2013) propose a set of three axioms, namely size, density and score monotonicity axioms, and check whether they are satisfied by eleven standard centrality measures but do not provide characterization results. Garg (2009) characterizes some centrality measures based on shortest paths. Kitti (2012) provides a characterization of eigenvector centrality without using consistency.

The closest paper to ours is the one by Palacios-Huerta and Volij (2004), who have used an axiomatic approach, and in particular a version of the consistency property, to measure the intellectual influence based on data on citations between scholarly publications. They find that the properties of invariance to reference intensity, weak homogeneity, weak consistency, and invariance to splitting of journals characterize a unique ranking method for the journals. Interestingly, this method, which they call the invariant method (Pinsky and Narin, 1976) is also at the core of the methodology used by Google to rank web sites (Page et al., 1998). The main difference with our approach is the way Palacios-Huerta and Volij (2004) define a reduced problem. In their paper, a reduced problem is non-embedded in the sense that it only contains nodes and links. As a consequence, they need to impose an ad hoc formula to split withdrawn initial links among the set of remaining nodes in the reduced problem. By contrast, the way we define an embedded and a reduced embedded network allows us to stick to a simpler and more common notion of reduction and to keep the same notion across characterizations.

Our focus on local centrality measures bears some resemblance with Echenique and Fryer (2007)'s emphasis on segregation indices that relate the segregation of an individual to the segregation of the individuals she interacts with. These authors propose a characterization of the "spectral segregation index" based on a linearity axiom that requires that one individual's segregation is a linear combination of her neighbors' segregation.

Finally, by providing an axiomatic characerization of Katz-Bonacich centrality, our paper complements Ballester et al. (2006) who provides its behavioral foundations; just as, for instance, Esteban and Ray (2011) complements Esteban and Ray (1994) for the concept of polarization. It makes Katz-Bonacich centrality one of the few economic concepts that possess both behavioral and axiomatic foundations.

The paper is organized as follows. In the next section, we recall some standard definitions related to networks and expose the concepts of *embedded* and *reduced embedded* networks. In section 3, we present our four main axioms. The first three, namely the *normalization, additivity* and *linearity* axioms, deal with behavior of the measure on very simple networks that we call *one-node embedded* networks. Those networks are star-networks and possess only one regular node. The fourth axiom is the consistency property. In section 4, we focus on the Katz-Bonacich centrality and prove our main characterization result (Proposition 1). In section 5, we present related axioms and extend the characterization result to degree centrality and eigenvector centrality. Finally, Section 6 concludes.

2 Definitions

2.1 Networks and Katz-Bonacich centrality

We consider a finite set of nodes $N = \{1, ..., n\}$. A *network* defined on N is a pair $(K, \mathbf{g})$ where $\mathbf{g}$ is a network on the set of nodes $K \subseteq N$. We adopt the *adjacency matrix* representation and denote by $k = |K|$, $\mathbf{g}$ is a $k \times k$ matrix with entry g_{ij} denoting whether node i is linked to node j. When node i is linked to node j in the network, $g_{ij} = 1$, otherwise $g_{ij} = 0$. The adjacency matrix is symmetric since we consider undirected links. Let $\mathcal{N}$ denote the finite set of networks defined on N.

The set of *neighbors* of a node i in network $(K, \mathbf{g})$ is denoted by $V_i(\mathbf{g})$. If we consider a subset of nodes $A \subseteq K$, the set $V_A(\mathbf{g})$ is the set of neighbors of the nodes in A that are not themselves in A, i.e. $V_A(\mathbf{g}) = \cup_{i \in A} V_i(\mathbf{g}) \cap \neg A$.

An *independent set* relative to network $(K, \mathbf{g})$ is a subset of nodes $A \subseteq K$ for which no two nodes are linked. A *dominating set* relative to network $(K, \mathbf{g})$ is a set of nodes $A \subseteq K$ such that every node not in A is linked to at least one node in A.

When we consider a network $(K, \mathbf{g})$, the k-square adjacency matrix $\mathbf{g}$ keeps track of the direct connections in the network. As is well known, the matrix $\mathbf{g}^p$, the pth power of $\mathbf{g}$, with coefficient $g_{ij}^{[p]}$, keeps track of the indirect connections in $(K, \mathbf{g})$: $g_{ij}^{[p]} \geq 0$ measures the number of paths of length $p \geq 1$ that go from i to j. By convention, $\mathbf{g}^0 = \mathbf{I}_k$, where $\mathbf{I}_k$ is the k-square identity matrix.

Given a sufficiently small scalar $a \geq 0$ and a network $(K, \mathbf{g})$, we define the matrix

$$\mathbf{M}(\mathbf{g}, a) \equiv [\mathbf{I}_k - a\mathbf{g}]^{-1} = \sum_{p=0}^{+\infty} a^p \mathbf{g}^p.$$

The parameter a is a decay factor that reduces the weight of longer paths in the right-hand-side sum. The coefficients $m_{ij}(\mathbf{g}, a) = \sum_{p=0}^{+\infty} a^p g_{ij}^{[p]}$ count the number of paths from i to j where paths of length p are discounted by a^p. Let also $\mathbf{1}_k$ be the k-dimensional vector of ones.

Definition 1 *The* **Katz-Bonacich centrality** *(Bonacich, 1987, Katz, 1953) is a function defined on $\mathcal{N}$ that assigns to every network $(K, \mathbf{g}) \in \mathcal{N}$ a k-dimensional vector of centralities defined as*

$$\mathbf{b}(\mathbf{g}, a) \equiv [\mathbf{I}_k - a\mathbf{g}]^{-1}\mathbf{1}_k, \tag{1}$$

where $0 \leq a < \frac{1}{n-1}$ for the matrix $\mathbf{M}(\mathbf{g}, a) \equiv [\mathbf{I}_k - a\mathbf{g}]^{-1}$ to be well-defined and nonnegative everywhere on $\mathcal{N}$.[7]

The Katz-Bonacich centrality of node i in $(K, \mathbf{g})$ is $b_i(\mathbf{g}, a) = \sum_{j \in K} m_{ij}(\mathbf{g}, a)$. It counts the number of paths from i to itself and the number of paths from i to any other node j. It is positive and takes values bigger than 1. Notice that, by a simple manipulation of equation (1), it is possible to define the vector of Katz-Bonacich centrality as a fixed point. For a in the relevant domain, it is the unique solution to the equation

$$\mathbf{b}(\mathbf{g}, a) = \mathbf{1}_k + a\,\mathbf{g}\,\mathbf{b}(\mathbf{g}, a). \tag{2}$$

[7]Theorems I^* and III^* in Debreu and Herstein (1953) ensure that $[\mathbf{I}_k - a\mathbf{g}]^{-1}$ exists and is nonnegative if and only if $a < \frac{1}{\lambda_{\max}}$ where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{g}$. Moreover, $\lambda_{\max}$ increases with the number of links in $\mathbf{g}$ and is maximal on $\mathcal{N}$ for the complete graph with n nodes where it takes value $n-1$.

According to this fixed-point formulation, the Katz-Bonacich centrality of node i depends exclusively on the centrality of its neighbors in $(K, \mathbf{g})$,

$$b_i(\mathbf{g}, a) = 1 + a \sum_{j \in K} g_{ij} b_j(\mathbf{g}, a) = 1 + a \sum_{j \in V_i(\mathbf{g})} b_j(\mathbf{g}, a).$$

2.2 Embedded networks and centrality measures

The following definitions are instrumental in the characterization of Katz-Bonacich centrality. We still consider a finite set of nodes N.

Definition 2 *An* **embedded network** *defined on N is a network in which nodes belong to one of two sets: the set of terminal nodes T and the set of regular nodes R, with $R \cup T \subseteq N$. The set of terminal nodes T forms an independent set and a real number $x_t \in \mathbb{R}^+$ is assigned to each terminal node $t \in T$. The set of regular nodes R forms a dominating set in $R \cup T$. An embedded network is therefore given by $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$, with $g_{tt'} = 0$ whenever $t, t' \in T$ and for all $t \in T$, $g_{tr} = 1$ for at least one $r \in R$.*

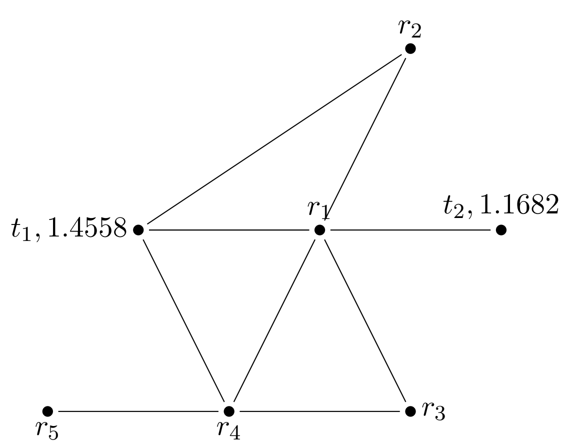

Figure 1: An Embedded Network $((\{r_1, r_2, r_3, r_4, r_5\} \cup \{t_1, t_2\}, \mathbf{g}), \{1.4558, 1.1682\})$

To illustrate this definition, consider the embedded network of Figure 1 which has five regular nodes and two terminal nodes. The terminal node t_1 is linked to three regular nodes, r_1, r_2 and r_4 and is assigned the positive number 1.4558. The terminal node t_2 is linked to a single regular node, r_1 and is assigned the positive number 1.1682.

Let $\bar{\mathcal{N}}$ denote the set of embedded networks defined on N. Of course, standard networks $(K, \mathbf{g})$ are embedded networks with $T = \emptyset$ and $\mathcal{N} \subset \bar{\mathcal{N}}$. A *one-node* *embedded network* is an embedded network that possesses exactly one regular node. It is therefore a star-shaped network in which all nodes except the center are assigned a real number. A *one-node network* is a one-node embedded network with $T = \emptyset$, it is therefore an isolated node. Figure 2 illustrates those two types of networks.

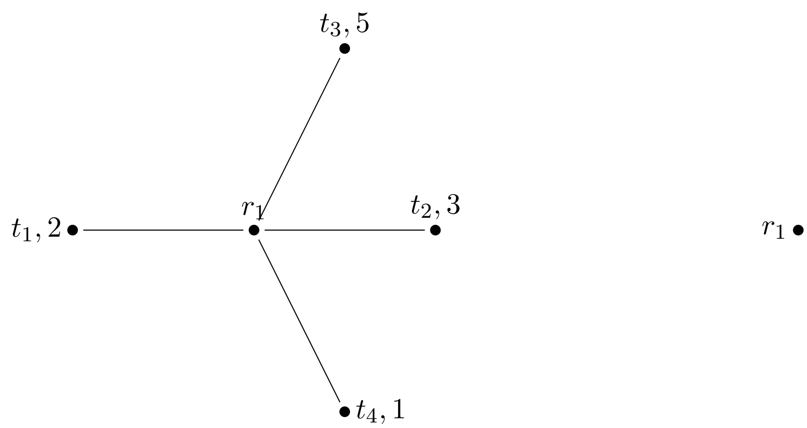

Figure 2: A One-Node Embedded Network (left) and a One-Node Network (right)

Definition 3 *A* **centrality measure** *defined on* $\bar{\mathcal{N}}$ *is a function that assigns to each embedded network* $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *in* $\bar{\mathcal{N}}$ *a* r*-dimensional vector of positive real numbers* $\mathbf{c} = (c_1, ..., c_r)$ *with* c_k *being the centrality of regular node* k*,* $k \in R$*.*

Observe that the centrality measure is only assigned to the regular nodes. Observe also that this definition adapts the usual notion of centrality to embedded networks. It is now possible to extend the definition of Katz-Bonacich centrality to any network in $\bar{\mathcal{N}}$

Definition 4 *A centrality measure defined on* $\bar{\mathcal{N}}$ *is a* **Katz-Bonacich centrality measure** *when there exists a positive scalar* a*,* $0 \leq a < \frac{1}{n-1}$*, such that it assigns to any embedded network* $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *in* $\bar{\mathcal{N}}$ *a* r*-dimensional vector of positive real numbers* $\mathbf{b}$ *that satisfy, for all* $i \in R$*,*

$$b_i = 1 + a \sum_{t \in V_i(\mathbf{g}) \cap T} x_t + a \sum_{j \in V_i(\mathbf{g}) \cap R} b_j.$$

According to this definition, the centrality of a node i is an affine combination of the real numbers assigned to its neighbors, either by the centrality measure itself or by the definition of the embedded network. When restricted to the domain $\mathcal{N}$, this definition coincides with the standard definition of Katz-Bonacich centrality as given in Section 2.1.

Definition 5 *Given any embedded network* $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *and any vector of real numbers* $(y_1, ..., y_r)$, $r = |R|$, *a* **reduced embedded network** *is an embedded network* $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})$ *where* $R' \subset R$, $T' = V_{R'}(\mathbf{g})$, $g'_{ij} = g_{ij}$ *when* $i \in R'$ *or* $j \in R'$ *and* $g'_{ij} = 0$ *when* $i, j \in T'$, *and* $x'_t = x_t$ *when* $t \in T$ *and* $x'_t = y_t$ *when* $t \in R$.

A reduced embedded network is constructed from an initial embedded network and a vector of real numbers. It keeps a subset of regular nodes in the initial embedded network together with their links. The new terminal nodes are the neighbors of this subset and they are assigned the real number they were assigned in the initial embedded network either via the vector x or the vector y.

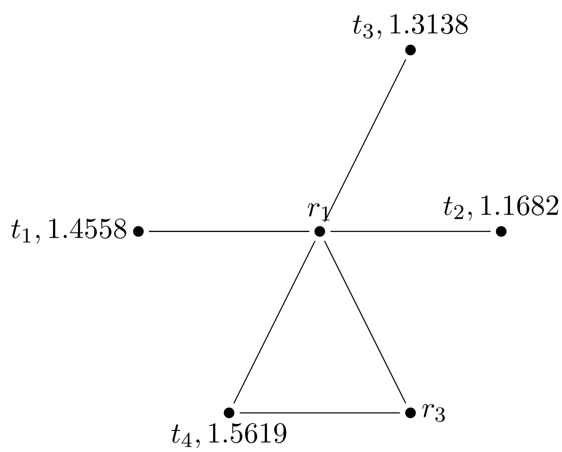

Figure 3: A Reduced Embedded Network obtained from Figure 1

To illustrate this definition, the reduced embedded network represented in Figure 3 is obtained from the network represented in Figure 1 together with the vector $\mathbf{y} = (1.6824, 1.3138, 1.3244, 1.5619, 1.1562)$ of positive numbers assigned respectively to $(r_1, r_2, r_3, r_4, r_5)$. In this reduced network, $R' = \{r_1, r_3\}$ and the new terminal nodes are $t_1, t_2, r_2 = t_3$ and $r_4 = t_4$. The real numbers assigned to terminal nodes t_1 and t_2 come from the initial embedded network while the real numbers assigned to terminal nodes t_3 and t_4 come from the vector $(1.6824, 1.3138, 1.3244, 1.5619, 1.1562)$.

3 Axioms

We start by listing some properties for a centrality measure on one-node embedded networks.

Axiom 1 *(***Normalization***) A centrality measure is* **(1,a)-normalized** *if and only if*

1. *for any one-node network* (i)*, the centrality measure of node* i *is* $c_i \equiv \bar{c} = 1$.
2. *for any one-node embedded network* $(i \cup j, g_{ij} = 1, x_j = 1)$*, the centrality of node* i *is* $c_i = \bar{c} + a = 1 + a$, $a \in \mathbb{R}^+$.

Nodes and links are the building blocks of networks. The normalization axiom provides information on the centrality of an isolated node and on the centrality of a node linked to a single terminal node to which is assigned the real number 1. It defines the centrality obtained from being alone as well as the centrality obtained from having one link.

Axiom 2 *(***Additivity***) Consider two one-node embedded networks* $((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *and* $((i \cup T', \mathbf{g}'), \{x_t\}_{t \in T'})$ *with* $T \cap T' = \emptyset$ *and with centrality measures* $\mathbf{c}$ *and* $\mathbf{c}'$*, the centrality measure is* **additive** *if and only if the centrality of the one-node embedded network* $((i \cup (T \cup T'), \mathbf{g} + \mathbf{g}'), \{x_t\}_{t \in T \cup T'})$[8] *is equal to* $\mathbf{c} + \mathbf{c}' - \bar{c}\mathbf{1}$*, i.e. the centrality of node* i *is* $c_i + c'_i - \bar{c}$.

This axiom says that if we start from two different one-node embedded networks (i.e. two star-shaped networks as in Figure 2) that have the same regular node (i.e. central node), then it suffices to add the contributions to the centrality of the regular node in each network to obtain the contribution to the centrality of the regular node in the one-node embedded network that "sums up" the two networks. Observe that the term $-\bar{c}$ in the formula above corresponds to the centrality of an isolated node. It is substracted from the sum of centralities in order not to count twice what node i brings in isolation.

[8]The notation $\mathbf{g} + \mathbf{g}'$ is used to describe the network that possesses the links of $\mathbf{g}$ and the links of $\mathbf{g}'$.

Axiom 3 *(**Linearity**) Consider the one-node embedded network* $((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *with centrality measure* $\mathbf{c}$*. The centrality measure is* **linear** *if and only if, for any* $\gamma > 0$*, the centrality measure of the one-node embedded network* $((i \cup T, \mathbf{g}), \{\gamma x_t\}_{t \in T})$ *is* $\bar{c}\mathbf{1} + \gamma(\mathbf{c} - \bar{c}\mathbf{1})$*, i.e. the centrality of node* i *is* $\bar{c} + \gamma(c_i - \bar{c})$*.*

This axiom says that, if we multiply by a positive parameter the values given to terminal nodes in a one-node embedded network, then the contribution to the centrality of the regular node (the central node) that comes from those terminal nodes is also multiplied by this positive parameter. Indeed, in the above formula, $c_i - \bar{c}$ corresponds to what being linked with the terminal nodes brings to the centrality of node i and $\bar{c}$ corresponds to what node i brings in isolation.

Axioms 1, 2 and 3 deal with properties of networks that possess exactly one regular node. The next axiom is key in extending the properties to any embedded network in $\bar{\mathcal{N}}$.

Axiom 4 *(**Consistency**) A centrality measure defined on* $\bar{\mathcal{N}}$ *is* **consistent** *if and only if for any embedded network* $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}) \in \bar{\mathcal{N}}$ *with centrality measure* $\mathbf{c} = (c_j)_{j \in R}$*, and for any reduced embedded network* $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})$ *where* $R' \subset R$*,* $T' = V_{R'}(\mathbf{g})$*, and* $x'_t = x_t$ *when* $t \in T \cap T'$ *and* $x'_t = c_t$ *when* $t \in R \cap T'$*, the centrality measure of the reduced embedded network is* $\mathbf{c} = (c_j)_{j \in R'}$*.*

The consistency property expresses the following idea. Suppose we start from an initial network and a vector of centralities and want to have a closer look at the centralities of a subset of nodes. We select this subset of nodes and compute again the centralities of the nodes in the reduced problem built from this subset of nodes and the initial vector of centralities. The measure is consistent if this computation leads to the same values for centralities as in the initial network.

Let us illustrate the consistency property (Axiom 4) with the networks of Figures 1 and 3. We need to assume that $a < 1/6 = 0.167$. Take, for example, $a = 0.1$. Consider the embedded network of Figure 1 where we assumed that $x_{t_1} = 1.4558$ and $x_{t_2} = 1.1682$. If we calculate the Katz-Bonacich centralities of all regular nodes, we easily obtain:

$$\begin{pmatrix} b_{r_1} \\ b_{r_2} \\ b_{r_3} \\ b_{r_4} \\ b_{r_5} \end{pmatrix} = \begin{pmatrix} 1.6824 \\ 1.3138 \\ 1.3244 \\ 1.5619 \\ 1.1562 \end{pmatrix}$$

Indeed, for node 1, we have:

$$\begin{aligned} b_{r_1} &= 1 + 0.1\,(x_{t_1} + x_{t_2}) + 0.1\,(b_{r_2} + b_{r_3} + b_{r_4}) \\ &= 1 + 0.1\,(1.4558 + 1.1682) + 0.1\,(b_{r_2} + b_{r_3} + b_{r_4}) \\ &= 1.2624 + 0.1\,(b_{r_2} + b_{r_3} + b_{r_4}) \end{aligned}$$

Similarly, for node 2, we have:

$$\begin{aligned} b_{r_2} &= 1 + 0.1\,(x_{t_1} + x_{t_2}) + 0.1 \times b_{r_1} \\ &= 1.2624 + 0.1 \times b_{r_1} \end{aligned}$$

In a similar way, we can calculate b_{r_3}, b_{r_4} and b_{r_5}. Then, by solving for these five equations for the five unknowns $b_{r_1}, b_{r_2}, b_{r_3}, b_{r_4}$ and b_{r_5}, we obtain the values of the Katz-Bonacich centralities shown above. Let us now calculate the Katz-Bonacich centralities of nodes 1 and 3 in the *reduced embedded network* (Figure 3). Assume $\mathbf{y} = \{1.6824, 1.3138, 1.3244, 1.5619, 1.1562\}$, which corresponds to the Katz-Bonacich centrality measures of nodes 1, 2, 3, 4 and 5 in Figure 1. Let us now check the *consistency* property, that is let us show that the Katz-Bonacich centralities of nodes 1 and 3 is the same in the embedded network and in the reduced embedded network. In the latter, we have:

$$\begin{aligned} b_{r_1} &= 1 + 0.1\,(x_{t_1} + x_{t_2} + x_{t_3} + x_{t_4}) + 0.1 \times b_{r_3} \\ &= 1 + 0.1\,(1.4558 + 1.1682 + 1.3138 + 1.5619) + 0.1 \times b_{r_3} \\ &= 1.55 + 0.1 \times b_{r_3} \end{aligned}$$

and

$$\begin{aligned} b_{r_3} &= 1 + 0.1 \times x_{t_4} + 0.1 \times b_{r_1} \\ &= 1 + 0.1 \times 1.5619 + 0.1 \times b_{r_1} \\ &= 1.1562 + 0.1 \times b_{r_1} \end{aligned}$$

By combining these two equations, it is straightforward to show that $b_{r_1} = 1.6824$ and $b_{r_3} = 1.3244$ and thus the Katz-Bonacich centralities are the same in both networks.

This is because, in y, we have chosen $x_{t_3} = b_{r_2} = 1.3138$ and $x_{t_4} = b_{r_4} = 1.5619$, where b_{r_2} and b_{r_4} have been calculated in the *embedded network* (Figure 1). Then it is clear that the Katz-Bonacich centralities of nodes 1 and 3 will be the same in the reduced embedded network and in the embedded network.

4 Characterization

We have the first following result:

Lemma 1 *A centrality measure defined on $\bar{\mathcal{N}}$ satisfies Axioms 1, 2 and 3 if and only if there exists an $a \in \mathbb{R}^+$ such that for any one-node embedded network $((i \cup T, \mathbf{g}), \{x_t\}_{t\in T})$, we have $\mathbf{c} = c_i = 1 + a\sum_{t\in T} x_t$.*

Proof : The *if* part of the proof is straightforward. For the *only if* part, take any one-node embedded network $((i \cup T, \mathbf{g}), \{x_l\}_{l\in T})$. Either $T = \emptyset$ and Axiom 1 ensures that the formula applies, or it can be constructed from a set of $|T|$ basic one-node embedded networks $((i \cup j, g_{ij} = 1, 1)$ which possess exactly one terminal node. In each of those basic networks, the normalization axiom ensures that $c_i = 1 + a$, the linearity axiom ensures that in any one-node network $((i \cup j, g_{ij} = 1), x_j)$, $c_i = 1 + ax_j$. Finally, by the additivity axiom, we know that in the initial one-node embedded network, $c_i = 1 + a\sum_{t\in T} x_t$. $\square$

Let us now state our main result:

Proposition 1 *A centrality measure defined on $\bar{\mathcal{N}}$ satisfies Axioms 1 to 4 if and only if it is a Katz-Bonacich centrality measure.*

Proof : (*If* part). It is straightforward to establish that a Katz-Bonacich centrality measure according to Definition 4 satisfies Axioms 1 to 4.

(*Only if* part). Axiom 4 and Lemma 1 imply that there exists a positive scalar a such that for any embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}) \in \bar{\mathcal{N}}$, the associated vector of centralities $\mathbf{c}$ satisfies, for all $i \in R$,

$$c_i = 1 + a \sum_{t\in V_i(\mathbf{g})\cap T} x_t + a \sum_{j\in V_i(\mathbf{g})\cap R} c_j. \quad (3)$$

In order to establish that $a < \frac{1}{n-1}$, consider the n-nodes complete network $(N, \mathbf{g}) \in \mathcal{N}$, for this network equation (3) can be written

$$c_i = 1 + a \sum_{j \neq i} c_j, \text{for all } i,$$

those c_i exist and are positive only when $a < \frac{1}{n-1}$. A centrality measure that satisfies Axioms 1 to 4 is therefore a Katz-Bonacich centrality measure. □

As shown by Ballester et al. (2006), the Katz-Bonacich centrality is closely related to the Nash equilibrium. Indeed, those authors show that in a game with quadratic payoffs and strategic complementarities played by agents located at the nodes of a network, the unique equilibrium actions are proportional to the Katz-Bonacich centralities of those nodes. In light of this important result, one can easily understand why, in the same vein as Peleg and Tijs (1996) who showed how consistency can be used to characterize the Nash equilibrium correspondence, it is possible to invoke consistency to characterize the Katz-Bonacich centrality measures.

In that case, characterization is further simplified because existence and uniqueness of the vector of centralities are guaranteed.

5 Extensions

In this section, we deal with two centrality measures that belong to the same class as the Katz-Bonacich centrality, i.e. the degree centrality and the eigenvector centrality.

5.1 Degree centrality

The degree centrality is one of the simplest centrality measures on networks. It assigns to each node a positive integer which corresponds to the number of neighbors this node possesses in the network. Formally, $d_i(\mathbf{g}) = |V_i(\mathbf{g})|$. It is well defined on $\bar{\mathcal{N}}$. We can slightly adapt our axioms to provide a characterization of degree centrality. Actually, the only changes concern the axioms that refer to one-node embedded networks.

Axiom 5 *(***Normalization***) The centrality measure is* **(0,1)-normalized** *if and only if*

1. *for any one-node network (i), the centrality measure of node i is $c_i \equiv \bar{c} = 0$.*

2. *for any one-node embedded network* $(i \cup j, g_{ij} = 1, x_j = 1)$*, the centrality of node* i *is* $c_i = 1$.

It is clear that we cannot use anymore Axiom 1 since the normalization is now different. In particular, an isolated node has a positive Katz-Bonacich centrality but a zero degree centrality.

Axiom 6 (**Invariance**) *Consider the one-node embedded network* $((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ *with centrality measure* $\mathbf{c}$*. The centrality measure is* **invariant** *if and only if, for any* $\gamma > 0$*, the centrality measure of the one-node embedded network* $((i \cup T, \mathbf{g}), \{\gamma x_t\}_{t \in T})$ *is* $\mathbf{c}$*, i.e. the centrality of node* i *is* c_i*.*

This axiom adapts the linearity axiom (Axiom 3) to the case of degree centrality. Clearly, for degree centrality, it does not matter if one multiplies by a positive parameter the positive values assigned to the terminal nodes.

Proposition 2 *A centrality measure defined on* $\bar{\mathcal{N}}$ *satisfies Axioms 2, 4, 5 and 6 if and only if it is the degree centrality measure.*

Proof : (*If* part). It is straightforward to establish that degree centrality satisfies those four Axioms.

(*Only if* part). Consider a centrality measure that satisfies Axioms 2, 5 and 6. Axiom 5 ensures that it assigns to any one-node network the real number 0 which is also its degree centrality. For any one-node embedded network that possess one terminal node and for which the real numbers assigned to the terminal node is equal to 1, the same Axiom 5 ensures that the centrality of the regular node is equal to its degree. Then, Axioms 2 and 6 ensure that the centrality of any one-node embedded network is its degree centrality. Finally, Axiom 4 implies that for any embedded network in $\bar{\mathcal{N}}$, the centrality measure assigns to each regular node its degree centrality. □

Katz-Bonacich and degree centralities are conceptually very close. They measure the centrality of one node by counting the paths that can be drawn from that node. In the case of degree centrality, attention is restricted to paths of length 1. In the case of the Katz-Bonacich centrality, all paths are considered. It is therefore not a surprise that their characterizations differ only marginally.

5.2 Eigenvector centrality

As highlighted in the introduction, the eigenvector centrality is a very important factor in explaining different outcomes. For example, the recent paper by Banerjee et al. (2013) shows that targeting individuals with the highest eigenvector centralities in a network of relationships would increase the adoption of a microfinance program by a substantial fraction of this population. Also, if we consider a network where journals are represented by nodes and references by links between those journals, then the eigenvector centrality seems to be a good way of ranking journals. For example, Pagerank (Brin and Page, 1998), which is closely related to eigenvector centrality, is the founding algorithm used by Google to sort its search results. Also, the measure eigenfactor (Bergstrom, 2007) uses Pagerank in order to assign different weights to each journal, and then it counts citations of each journal weighting them by the Pagerank of the source.

More precisely, eigenvector centrality is a measure that builds upon properties of nonnegative square matrices. To each node i in a network $(K, \mathbf{g})$, the eigenvector centrality assigns a positive real number c_i that is proportional to the sum of the centralities of its neighbors so that there exists a positive λ satisfying, for all $i \in K$,

$$\lambda c_i = \sum_{j \in V_i(\mathbf{g})} c_j.$$

Written in matrix form and denoting $\mathbf{c}$ the k-dimensional vector of centralities, we have:

$$\lambda \mathbf{c} = \mathbf{g}\mathbf{c}. \tag{4}$$

This formula highlights the fact that λ is an eigenvalue of $\mathbf{g}$, that $\mathbf{c}$ is a corresponding eigenvector and therefore that the c_i are defined up to a multiplicative constant. The Perron-Frobenius theorem ensures that all the c_i can be chosen positive when $\lambda \geq 0$ is the largest eigenvalue of $\mathbf{g}$. Moreover, if $(K, \mathbf{g})$ is a connected network, i.e. a network such that all pairs of nodes are path-connected, then requiring that all the c_i are positive implies that λ is necessarily the largest eigenvalue of $\mathbf{g}$.

In order to provide an axiomatic characterization of the eigenvector centrality along the same lines as we did for the Katz-Bonacich and the degree centrality, we need to adapt some of our definitions, in particular, because of the potential multiplicity of acceptable eigenvalues and eigenvectors, we now deal with correspondences and not with functions.

Definition 6 *A* **centrality correspondence** *defined on $\bar{\mathcal{N}}$ is a correspondence ϕ that assigns to each embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$ in $\bar{\mathcal{N}}$ a set of $(r+1)$-dimensional vectors of positive real numbers $(\lambda, \mathbf{c}) = (\lambda, c_1, ..., c_r)$ with c_k being the centrality of regular node k, $k \in R$.*

Definition 7 *The* **eigenvector centrality correspondence** *ϕ^e defined on $\bar{\mathcal{N}}$ assigns to any embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$ in $\bar{\mathcal{N}}$ the set of $(r+1)$-dimensional vectors of positive real numbers $(\lambda, \mathbf{c})$ that satisfy for all $i \in R$*

$$\lambda c_i^e = \sum_{j \in V_i(\mathbf{g}) \cap R} c_j^e + \sum_{t \in V_i(\mathbf{g}) \cap T} x_t.$$

Axiom 7 *(***Normalization***) The centrality correspondence ϕ is* **normalized** *if and only if*

1. *for any one-node network (i), $\phi(i) = \{(\lambda, c) : c \in \mathbb{R}^+, \lambda \in \mathbb{R}^+, \lambda c = 0\}$,*

2. *for any one-node embedded network $(i \cup j, g_{ij} = 1, x_j = 1)$, $\phi(i \cup j, g_{ij} = 1, x_j = 1) = \{(\lambda, c) : c \geq 0$ and $\lambda\, c = 1\}$.*

Axiom 8 *(***Additivity***) Consider two one-node embedded networks $((i \cup T, \mathbf{g}), \{x_t\}_{t\in T})$ and $((i \cup T', \mathbf{g}'), \{x_t\}_{t\in T'})$ with $T \cap T' = \emptyset$, the centrality correspondence ϕ is* **additive** *if and only if*

$$\phi(((i \cup (T \cup T'), \mathbf{g} + \mathbf{g}'), \{x_t\}_{t \in T \cup T'})) = \{(\lambda, c) : \exists c_i, c_i' \text{ such that } c = c_i + c_i',$$

$$(\lambda, c_i) \in \phi(((i \cup T, \mathbf{g}), \{x_t\}_{t\in T})) \text{ and } (\lambda, c_i') \in \phi(((i \cup T', \mathbf{g}'), \{x_t\}_{t\in T'}))\}.$$

Axiom 9 *(***Linearity***) Consider the one-node embedded network $((i \cup T, \mathbf{g}), \{x_t\}_{t\in T})$. The centrality correspondence is* **linear** *if and only if, for any $\gamma > 0$,*

$$\phi(((i \cup T, \mathbf{g}), \{\gamma x_t\}_{t\in T})) = \{(\lambda, c) : (\lambda, \frac{c}{\gamma}) \in \phi(((i \cup T, \mathbf{g}), \{x_t\}_{t\in T}))\}.$$

Axiom 10 *(***Consistency***) A centrality correspondence defined on $\bar{\mathcal{N}}$ is* **consistent** *if and only if for any embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}) \in \bar{\mathcal{N}}$, any vector $(\lambda, \mathbf{c}) \in \phi(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$, and any reduced embedded network $((R' \cup T', \mathbf{g}'), \{x_t'\}_{t\in T'})$ where $R' \subset R$, $T' = V_{R'}(\mathbf{g})$, and $x_t' = x_t$ when $t \in T \cap T'$ and $x_t' = c_t$ when $t \in R \cap T'$, we have $(\lambda, (c_i)_{i\in R'}) \in \phi(((R' \cup T', \mathbf{g}'), \{x_t'\}_{t\in T'}))$.*

Because we are now dealing with correspondences and the consistency property is written in terms of set inclusions, those four axioms no longer characterize a unique centrality and we need to invoke an additional property. Consider the correspondence $\tilde{\phi}$ where, for any embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$, $\tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$ is defined as the set of $(\lambda, \mathbf{c})$ such that for any reduced embeded network $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t\in T'})$ constructed from $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$ and that vector $(\lambda, \mathbf{c})$, we have

$$(\lambda, (c_i)_{i\in R'}) \in \phi(((R' \cup T', \mathbf{g}'), \{x'_t\}_{t\in T'})).$$

Axiom 11 *(***Converse consistency***) A centrality correspondence defined on $\bar{\mathcal{N}}$ is* **converse consistent** *if and only if for any embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}) \in \bar{\mathcal{N}}$,*

$$\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})) \supseteq \tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})).$$

The term converse consistency is easily understood when one realizes that consistency is equivalent to $\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})) \subseteq \tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t\in T}))$.

Proposition 3 *A centrality correspondence defined on $\bar{\mathcal{N}}$ satisfies Axioms 7, 8, 9, 10 and 11 if and only if it is the eigenvector centrality correspondence.*

Proof: (*If* part). Verifying that the eigenvector centrality correspondence ϕ^e satisfies Axioms 7, 8, 9 is straightforward. Then consider the correpondence $\tilde{\phi}^e$. By construction, it assigns to any embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t\in T})$ the set of $(r+1)$-dimensional vectors of positive real numbers $(\lambda, \mathbf{c})$ that satisfy for all $i \in R$

$$\lambda c_i = \sum_{j\in V_i(\mathbf{g})\cap R} c_j + \sum_{t\in V_i(\mathbf{g})\cap T} x_t.$$

In other words, $\tilde{\phi}^e = \phi^e$ and the eigenvector centrality correspondence satisfies Axioms 10 and 11.

(*Only if* part). The proof is by induction on the number of regular nodes.

Initializing: Consider a centrality correspondence ϕ that satisfies Axioms 7, 8, 9. For any one-node network (i), $\phi(i) = \{(\lambda, c) : \lambda \in \mathbb{R}^+, c \in \mathbb{R}^+, \lambda c = 0\}$ by Axiom 7. For any one-node embedded network $((i \cup j, g_{ij} = 1), x_j)$, Axioms 7 and 9 imply that

$$\phi(((i \cup j, g_{ij} = 1), x_j)) = \{(\lambda, c_i) : \ \lambda \in \mathbb{R}^+, \ c_i \in \mathbb{R}^+, \ \lambda c_i = x_j\}.$$

Then, Axiom 8 implies that for any one-node embedded network $((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})$,

$$\phi(((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})) = \{(\lambda, c_i) : \lambda \in \mathbb{R}^+,\ c_i \in \mathbb{R}^+,\ \lambda c_i = \sum_{t \in T} x_t\}.$$

Therefore, $\phi(((i \cup T, \mathbf{g}), \{x_t\}_{t \in T})) = \phi^e(((i \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$, and ϕ coincides with the eigenvector centrality correspondence on the set of one-node embedded networks.

Induction hypothesis: A centrality correspondence ϕ that satisfies Axioms 7, 8, 9, 10 and 11 coincides with the eigenvector centrality correspondence for any embedded network that possesses at most $r-1$ regular nodes.

Induction step: Consider an embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$ that possesses r regular nodes and a centrality correspondence ϕ that satisfies Axioms 7, 8, 9, 10 and 11. Axiom 10 together with the induction hypothesis imply that for any vector $(\lambda, \mathbf{c}) \in \phi(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$, and any node $i \in R$,

$$\lambda c_i = \sum_{j \in V_i(\mathbf{g}) \cap R} c_j + \sum_{t \in V_i(\mathbf{g}) \cap T} x_t,$$

i.e. imply that $\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})) \subseteq \phi^e(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$.

The induction hypothesis implies that, for any embedded network $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$, $\tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$ is defined as the set of $(\lambda, \mathbf{c})$ such that, for any reduced embeded network $((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})$ constructed from $((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})$, we have

$$(\lambda, (c_i)_{i \in R'}) \in \phi^e(((R' \cup T', \mathbf{g}'), \{x'_t\}_{t \in T'})).$$

This implies that $\tilde{\phi}(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})) \supseteq \phi^e(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$. Then Axiom 11 implies that $\phi(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T})) \supseteq \phi^e(((R \cup T, \mathbf{g}), \{x_t\}_{t \in T}))$.

Therefore we conclude that ϕ coincides with ϕ^e on the set of embedded networks with at most r regular nodes. □

Despite the fact that the spectral properties of a network may not be invariant under the reduction operation, we thus show that it is possible to characterize eigenvector centrality with a small set of simple axioms that includes the consistency property.

6 Conclusion

In this paper, we propose an axiomatic approach to derive a class of centrality measures for which the centrality of an agent is recursively related to the centralities of agents she is connected to. This includes the Katz-Bonacich, the degree and the eigenvector centrality. The core of our argument is based on the consistency axiom, which relates the properties of the measure for a given network to its properties for a reduced problem. In our case, the reduced problem only keeps tract of *local* and parsimonious information. This is possible because all the centralities study here are local in the sense that the centrality measure of an agent only depends on the centrality measure of her neighbors.

We believe that our methodology could be extended to other centrality measures such as the closeness centrality. This measure (and many others) is based on shortest paths and it is possible to compute the shortest paths from any node in a network from the shortest paths of all her neighbors. However, this means that, in order to derive the axioms for the closeness centrality, we would need much more information than only the centrality of the neighbors. We would need to know all the shortest paths stemming from the neighbors. The definition of embedded and reduced embedded networks would need to be adapted accordingly. This is clearly an interesting project that we leave for future research.

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