Network Games with Incomplete Information*

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Abstract

We consider a network game with strategic complementarities where the individual reward or the strength of interactions is only partially known by the agents. Players receive different correlated signals and they make inferences about other players’ information. We demonstrate that there exists a unique Bayesian-Nash equilibrium. We characterize the equilibrium by disentangling the information effects from the network effects and show that the equilibrium effort of each agent is a weighted combinations of different Katz-Bonacich centralities where the decay factors are the eigenvalues of the information matrix while the weights are its eigenvectors. We then study the impact of incomplete information on a network policy which aim is to target the most relevant agents in the network (key players). Compared to the complete information case, we show that the optimal targeting may be very different.

Keywords: social networks, strategic complementarities, Bayesian games, key player policy.

JEL Classification: C72, D82, D85.

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1 Introduction

Social networks are important in numerous facets of our lives. For example, the decision of an agent to buy a new product, attend a meeting, commit a crime, find a job is often influenced by the choices of his or her friends and acquaintances. The emerging empirical evidence on these issues motivates the theoretical study of network effects. For example, job offers can be obtained from direct and indirect acquaintances through word-of-mouth communication. Also, risk-sharing devices and cooperation usually rely on family and friendship ties. Spread of diseases, such as AIDS infection, also strongly depends on the geometry of social contacts. If the web of connections is dense, we can expect higher infection rates.

Network analysis is a growing field within economics\(^1\) because it can analyze the situations described above and provides interesting predictions in terms of equilibrium behavior. A recent branch of the network literature has focused on how network structure influences individual behaviors. This is modeled by what are sometimes referred to as “games on networks” or “network games”.\(^2\) The theory of “games on networks” considers a game with \(n\) agents (that can be individuals, firms, regions, countries, etc.) who are embedded in a network. Agents choose actions (e.g., buying products, choosing levels of education, engaging in criminal activities, investing in R&D, etc.) to maximize their payoffs, given how they expect others in their network to behave. Thus, agents implicitly take into account interdependencies generated by the social network structure. An important paper in this literature is that of Ballester et al. (2006). They compute the Nash equilibrium of a network game with strategic complementarities when agents choose their efforts simultaneously. In their setup, restricted to linear-quadratic utility functions, they establish that, for any possible network, the peer effects game has a unique Nash equilibrium where each agent effort’s is proportional to her Katz-Bonacich centrality measure. This is a measure introduced by Katz (1953) and Bonacich (1987), which counts all paths starting from an agent but gives a smaller value to connection that are farther away.

While settings with a fixed network are widely applicable, there are also many applications where players choose actions without fully knowing with whom they will interact. For example, learning a language, investing in education, investing in a software program, and so forth. These can be better modeled using the machinery of incomplete information games. This is what we do in this paper.

To be more precise, we consider a model similar to that of Ballester et al. (2006) but

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\(^2\)For a recent overview of this literature, see Jackson and Zenou (2015).
where the individual reward or the strength of interactions is partially known. In other words, this is a model where the state of world (i.e. the marginal return of effort or the synergy parameter) is common to all agents but only partially known by them. We assume that there is no communication between the players and that the network does not affect the possible channels of communication between them.

We start with a simple model with imperfect information on the marginal return of effort and where there are two states of the world. All individuals share a common prior and each individual receives a private signal, which is partially informative. Using the same condition as in the perfect information case, we show that there exists a unique Bayesian-Nash equilibrium. We can also characterize the Nash equilibrium of this game for each agent and for each signal received by disentangling the network effects from the information effects by showing that each effort is a weighted combination of two Katz-Bonacich centralities where the decay factors are the eigenvalues of the information matrix times the synergy parameter while the weights involve conditional probabilities, which include beliefs about the states of the world given the signals received by all agents. We then extend our model to any number of the states of the world and any signal. We demonstrate that there also exists a unique Bayesian-Nash equilibrium and give a complete characterization of equilibrium efforts as a function of weighted Katz-Bonacich centralities and information aspects. We also derive similar results for the case when the strength of interactions is partially known.

We then study a policy analyzed by Ballester et al. (2006, 2010). The aim is to solve the planner’s problem that consists in finding and getting rid of the key player, i.e., the agent who, once removed, leads to the highest reduction in aggregate activity. If the planner has incomplete information about the marginal return of effort (or the strength of interactions), then we show that the key player may be different to the one proposed in the perfect information case. This difference is determined by a ratio that captures all the (imperfect) information that the agents have, including the priors of the agents and the planner and the posteriors of the agents.

The paper unfolds as follows. In the next section, we relate our paper to the network literature with incomplete information. In Section 3, we characterize the equilibrium in the model with perfect information and show under which condition there exists a unique Nash equilibrium. Section 4 deals with a simple model with only two states of the world and two signals when the marginal return of effort is partially known. In Section 5, we analyze the general model when there is a finite number of states of the world and signals for the case when the marginal return of effort is unknown. In Section 6, we discuss the case when

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3See Zhou and Chen (2015) who introduce the concept of key leader, which is related to that of key player.
the information matrix is not diagonalizable. Section 7 analyzes the key player policy when information is complete and incomplete. Finally, Section 8 concludes. Throughout the paper, we will use the same two examples to illustrate all our results. The main assumptions of the model and their implications are stated in Appendix A.1. In Appendix A.2, we analyze the general model when there is a finite number of states of the world and signals for the case when the strength of interactions is unknown. In Appendix A.3, we provide some results for the Kronecker product used in the paper. Appendix A.4 deals with case when the information matrix is not diagonalizable and where we resort to the Jordan decomposition. The proofs of all lemmas and propositions in the main text can be found in Appendix A.5.

2 Related literature

Our paper is a contribution to the literature on “games on networks” or “network games” (Jackson and Zenou, 2015). We consider a game with strategic complements where an increase in the actions of other players leads a given player’s higher actions to have relatively higher payoffs compared to that player’s lower actions. In this framework, we consider a game with imperfect information on either the marginal payoff of effort or the strength of interaction. There is a relatively small literature which looks at the issue of imperfect information in this class of games.

Galeotti et al. (2010) and Jackson and Yariv (2007) are related to our paper but they study a very different dimension of uncertainty—i.e., uncertainty about the network structure. They show that an incomplete information setting can actually simplify the analysis of games on networks. In particular, results can be derived showing how agents’ actions vary with their degree.4

There is also an interesting literature on learning in networks. Bala and Goyal (1998) were among the first to study this issue and show that each agent in a connected network will obtain the same long-run utility and that, if the network is large enough and there are enough agents who are optimistic about each action spread throughout the network, then the probability that the society will converge to the best overall action can be made arbitrarily close to 1. More recently, Acemoglu et al. (2011) study a model where the state of the world is unknown and affects the action and the utility function of each agent. Each agent forms beliefs about this state from a private signal and from her observation of the actions of other agents. As in our model, agents can update their beliefs in a Bayesian way. They show that when private beliefs are unbounded (meaning that the implied likelihood ratios

4See Jackson and Yariv (2011) for an overview of this literature.
are unbounded), there will be asymptotic learning as long as there is some minimal amount of “expansion in observations”\textsuperscript{5}.

Compared to the literature of incomplete information in networks, our paper is the first to consider a model with a common unknown state of the world (i.e. the marginal return of effort or the synergy parameter), which is partially known by the agents and where there is neither communication nor learning\textsuperscript{6}. We first show that there exists a unique Bayesian-Nash equilibrium. We are also able to completely characterize this unique equilibrium. This characterization is such that each equilibrium effort is a combination of different Katz-Bonacich centralities, where the decay factors are the corresponding eigenvalues of the information matrix while the weights are the elements of matrices that have eigenvectors as columns. We are able to do so because we could diagonalize both the adjacency matrix of the network, which lead to the Katz-Bonacich centralities, and the information matrix. We believe that this is one of our main results and has not been done before.

3 The complete information case

3.1 The model

The network Let \( I := \{1, \ldots, n\} \) denote the set of players, where \( n > 1 \), connected by a network \( g \). We keep track of social connections in this network by its symmetric adjacency matrix \( G = [g_{ij}] \), where \( g_{ij} = g_{ji} = 1 \) if \( i \) and \( j \) are linked to each other, and \( g_{ij} = 0 \), otherwise. We also set \( g_{ii} = 0 \). The reference group of individual \( i \) is the set of \( i \)'s neighbors given by \( N_i = \{ j \neq i \mid g_{ij} = 1 \} \). The cardinal of the set \( N_i \) is \( g_i = \sum_{j=1}^{n} g_{ij} \), which is known as the degree of \( i \) in graph theory.

Payoffs Each agent takes action \( x_i \in [0, +\infty) \) that maximizes the following quadratic utility function:

\[
    u_i (x_i, x_{-i}; G) = \alpha x_i - \frac{1}{2} x_i^2 + \beta \sum_{j=1}^{n} g_{ij} x_i x_j
\]

where \( \alpha \) is the marginal return of effort and \( \beta \) is the strength of strategic interactions (synergy parameter).

\textsuperscript{5}For overviews on these issues, see Jackson (2008, 2011) and Goyal (2011).

\textsuperscript{6}Bergemann and Morris (2013) propose an interesting paper on these issues but without an explicit network analysis. Blume et al. (2015) develop a network model with incomplete information but mainly focus on identification issues.
The first two terms of the utility function corresponds to a standard cost-benefit analysis without the influence of others. In other words, if individual $i$ was isolated (not connected in a network), she will choose the optimal action $x_i^* = \alpha$, independent of what the other agents choose. The last term in (1) reflects the network effects, i.e. the impact of the agents’ links aggregate effort levels on $i$’s utility. As agents may have different locations in a network and their friends may choose different effort levels, the term $\sum_{j=1}^{n} g_{ij} x_i x_j$ is heterogeneous in $i$. The coefficient $\beta$ captures the local-aggregate endogenous peer effect. More precisely, bilateral influences for individual $i, j$ ($i \neq j$) are captured by the following cross derivatives

$$\frac{\partial^2 u_i (x_i, x_{-i}; G)}{\partial x_i \partial x_{j}} = \beta g_{ij}$$

As we assume $\beta > 0$, if $i$ and $j$ are linked, the cross derivative is positive and reflects strategic complementarity in efforts. That is, if $j$ increases her effort, then the utility of $i$ will be higher if $i$ also increases her effort. Furthermore, the utility of $i$ increases with the number of friends.

In equilibrium, each agent maximizes her utility (1). From the first-order condition, we obtain the following best-reply function for individual $i$

$$x_i^* = \alpha + \beta \sum_{j=1}^{n} g_{ij} x_j^*$$

The Katz-Bonacich network centrality measure Let $G^k$ be the $k$th power of $G$, with coefficients $g_{ij}^{[k]}$, where $k$ is some nonnegative integer. The matrix $G^k$ keeps track of the indirect connections in the network: $g_{ij}^{[k]} \geq 0$ measures the number of walks of length $k \geq 1$ in $g$ from $i$ to $j$. In particular, $G^0 = I_n$, where $I_n$ is the $n \times n$ identity matrix.

Denote by $\lambda_{\text{max}} (G)$ the largest eigenvalue of $G$.

**Definition 1** Consider a network $g$ with adjacency $n$-square matrix $G$ and a scalar $\beta > 0$ such that $\beta < 1/\lambda_{\text{max}} (G)$.

(i) Given a vector $u_n \in \mathbb{R}^n_+$, the vector of $u_n$-weighted Katz-Bonacich centralities of parameter $\beta$ in $g$ is:

$$b_{u_n} (\beta, G) := \sum_{k=0}^{+\infty} \beta^k G^k u_n = (I_n - \beta G)^{-1} u_n$$

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7A walk of length $k$ from $i$ to $j$ is a sequence $\langle i_0, \ldots, i_k \rangle$ of players such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and $g_{i_p i_{p+1}} > 0$, for all $0 \leq p \leq k - 1$, that is, players $i_p$ and $i_{p+1}$ are directly linked in $g$. In fact, $g_{ij}^{[k]}$ accounts for the total weight of all walks of length $k$ from $i$ to $j$. When the network is un-weighted, that is, $G$ is a $(0,1)$-matrix, $g_{ij}^{[k]}$ is simply the number of paths of length $k$ from $i$ to $j$. 

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(ii) If \( \mathbf{u}_n = \mathbf{1}_n \), where \( \mathbf{1}_n \) is the \( n \)-dimensional vector of ones, then the \textit{unweighted} Katz-Bonacich centrality of parameter \( \beta \) in \( g \) is:

\[
\mathbf{b}(\beta, \mathbf{G}) := \sum_{k=0}^{+\infty} \beta^k \mathbf{G}^k \mathbf{1}_n = (\mathbf{I}_n - \beta \mathbf{G})^{-1} \mathbf{1}_n
\]

If we consider the \textit{unweighted} Katz-Bonacich centrality of node \( i \) (defined by (5)), i.e. \( b_i(\beta, \mathbf{G}) \), it counts the \textit{total} number of walks in \( g \) starting from \( i \) and discounted by distance. By definition, \( b(\beta, \mathbf{G}) \geq 1 \), with equality when \( \beta = 0 \). The \( \mathbf{u}_n \)-weighted Katz-Bonacich centrality of node \( i \) (defined by (4)), i.e. \( b_{i,u}(\beta, \mathbf{G}) \), has a similar interpretation with the additional fact that the walks have to be weighted by the vector \( \mathbf{u}_n \).

We have a first result due to Ballester et al. (2006).

**Proposition 1** If \( \alpha > 0 \) and \( 0 < \beta < 1/\lambda_{\text{max}}(\mathbf{G}) \), then the network game with payoffs (1) has a unique interior Nash equilibrium in pure strategies given by

\[
x_i^* = \alpha b_i(\beta, \mathbf{G})
\]

The equilibrium Katz-Bonacich centrality measure \( b(\beta, \mathbf{G}) \) is thus the relevant network characteristic that shapes equilibrium behavior. This measure of centrality reflects both the direct and the indirect network links stemming from each agent.

To understand why the above characterization in terms of the largest eigenvalue \( \lambda_{\text{max}}(\mathbf{G}) \) works, and to connect the analysis to what follows below, we provide now a characterization of the solution using the fact that the adjacency matrix \( \mathbf{G} \) is diagonalizable. The system that characterizes the equilibrium of the game (6) is:

\[
x^* = \alpha (\mathbf{I}_n - \beta \mathbf{G})^{-1} \mathbf{1}_n
\]

To resolve this system we are going to diagonalize \( \mathbf{G} \), which is assumed to be symmetric and thus diagonalizable. We have that \( \mathbf{G} = \mathbf{C} \mathbf{D}_G \mathbf{C}^{-1} \), where \( \mathbf{D}_G \) is a \( n \times n \) diagonal matrix with entries equal to the eigenvalues of the matrix \( \mathbf{G} \), i.e.

\[
\mathbf{D}_G = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_n
\end{pmatrix}
\]
where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( G \). The Neuman series \( \sum_{k=0}^{+\infty} \beta^k (D_G)^k \) converges if and only if \( \beta < 1/\lambda_{\text{max}}(G) \). If it converges, then \( \sum_{k=0}^{+\infty} \beta^k (D_G)^k = (I_n - \beta G)^{-1} \). In such case, we have that

\[
(I_n - \beta G)^{-1} = \sum_{k=0}^{+\infty} \beta^k G^k = C \left[ \sum_{k=0}^{+\infty} \beta^k (D_G)^k \right] C^{-1}
\]

Observe that the “if and only if condition” is due to the fact that the diagonal entries of \( \sum_{k=0}^{+\infty} \beta^k (D_G)^k \) are power series of rates equal to \( \beta \lambda_i (G) \), and all these power series converge if and only if \( \beta \lambda_{\text{max}}(G) < 1 \), which is equivalent to the condition written in Proposition 1. These terms are obviously very easy to compute. Indeed, they are equal to \( \frac{1}{1 - \beta \lambda_i(G)} \) for each \( i \in \{1, \ldots, n\} \). The off-diagonal elements of \( \sum_{k=0}^{+\infty} \beta^k (D_G)^k \) are all equal to 0.

### 3.2 Example 1

Consider the network \( g \) in Figure 1 with three players, where agent 1 holds a central position whereas agents 2 and 3 are peripherals.

![Figure 1: Star network](image)

The adjacency matrix for this network is the following:\(^9\)

\[
G^S = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

Assume that \( \alpha = 1 \). We can now compute the players’ centrality measures. We obtain:\(^{10}\)

\[
b_1 (\beta, G^S) = \sum_{k=0}^{+\infty} \left[ \beta^{2k} 2^k + \beta^{2k+1} 2^{k+1} \right] = \frac{1 + 2\beta}{1 - 2\beta^2}
\]

\[
b_2 (\beta, G^S) = b_3 (\beta, G) = \sum_{k=0}^{+\infty} \left[ \beta^{2k} 2^k + \beta^{2k+1} 2^k \right] = \frac{1 + \beta}{1 - 2\beta^2}
\]

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\(^9\)The superscript \( S \) refers to the star network.

\(^{10}\)Note that these centrality measures are only well-defined when \( \beta < 1/\sqrt{2} \) (condition on the largest eigenvalue).
According to intuition, player 1 has the highest centrality measure. All centrality measures $b_i$s increase with $\beta$, and so does the ratio $b_1/b_2$ of agent 1’s centrality with respect to any other agent, as the contribution of indirect walks to centrality increases with $\beta$. We obtain the following efforts at equilibrium (Proposition 1):

$$x^*_1 = \alpha \left( \frac{1 + 2\beta}{1 - 2\beta^2} \right)$$

(7)

and

$$x^*_2 = x^*_3 = \alpha \left( \frac{1 + \beta}{1 - 2\beta^2} \right)$$

(8)

As expected, the effort exerted by player 1, the most central player, is the highest one.

4 The incomplete information case: A simple model when $\alpha$ is unknown

We develop a simple model with common values and private information where there are only two states of the world and two signals.

4.1 The model

Assume that the marginal return of effort $\alpha$ in the payoff function (1) is common to all agents but only partially known by the agents. Agents know, however, the exact value of the synergy parameter $\beta$.$^{11}$

**Information** We assume that there are two states of the world, so that the parameter $\alpha$ can only take two values: $\alpha_l < \alpha_h$. All individuals share a common prior:

$$\mathbb{P}(\{\alpha = \alpha_h\}) = p \in (0, 1)$$

Each individual $i$ receives a private signal, $s_i \in \{h, l\}$, such that

$$\mathbb{P}(\{s_i = h\} | \{\alpha = \alpha_h\}) = \mathbb{P}(\{s_i = l\} | \{\alpha = \alpha_l\}) = q \geq 1/2$$

where $\{s_i = h\}$ and $\{s_i = l\}$ denote, respectively, the event that agent $i$ has received a signal $h$ and $l$. Assume that there is no communication between the players and that the network does not affect the possible channels of communication between them. Denote by

$$\mathbb{P}(\{s_i = l\}) = \mathbb{P}(\{s_i = l\} | \{\alpha = \alpha_l\}) \mathbb{P}(\{\alpha = \alpha_l\}) + \mathbb{P}(\{s_i = l\} | \{\alpha = \alpha_h\}) \mathbb{P}(\{\alpha = \alpha_h\})$$

$^{11}$We consider the case of unknown $\beta$ in Appendix A.2.
and
\[ \mathbb{P} (\{s_i = h\}) = \mathbb{P} (\{s_i = h\} | \{\alpha = \alpha_l\}) \mathbb{P} (\{\alpha = \alpha_l\}) + \mathbb{P} (\{s_i = h\} | \{\alpha = \alpha_h\}) \mathbb{P} (\{\alpha = \alpha_h\}) \]

Then, if agent \( i \) receives the signal \( s_i = l \), using the Bayes’ rule, we have:
\[ \mathbb{P} (\{\alpha = \alpha_l\} | \{s_i = l\}) = \frac{\mathbb{P} (\{s_i = l\} | \{\alpha = \alpha_l\}) \mathbb{P} (\{\alpha = \alpha_l\})}{\mathbb{P} (\{s_i = l\})} = \frac{q (1 - p)}{q (1 - p) + (1 - q) p} \quad (9) \]

and
\[ \mathbb{P} (\{\alpha = \alpha_h\} | \{s_i = l\}) = 1 - \mathbb{P} (\{\alpha = \alpha_l\} | \{s_i = l\}) = \frac{(1 - q) p}{q (1 - p) + (1 - q) p} \quad (10) \]

Similarly, if agent \( i \) receives the signal \( h \), then using the Bayes’ rule, we have:
\[ \mathbb{P} (\{\alpha = \alpha_l\} | \{s_i = h\}) = \frac{\mathbb{P} (\{s_i = h\} | \{\alpha = \alpha_l\}) \mathbb{P} (\{\alpha = \alpha_l\})}{\mathbb{P} (\{s_i = h\})} = \frac{(1 - q) (1 - p)}{(1 - q) (1 - p) + qp} \quad (11) \]

and
\[ \mathbb{P} (\{\alpha = \alpha_h\} | \{s_i = h\}) = 1 - \mathbb{P} (\{\alpha = \alpha_l\} | \{s_i = h\}) = \frac{qp}{(1 - q) (1 - p) + qp} \quad (12) \]

The Bayesian Game  Given that there is incomplete information about the state of the world \( \alpha \) and about others’ information, this is a Bayesian game. Agent \( i \) has to choose an action \( x_i (s_i) \geq 0 \) for each signal \( s_i \in \{l, h\} \). The expected utility of agent \( i \) can be written as:
\[ \mathbb{E} [u_i | s_i] = \mathbb{E} [\alpha | s_i] x_i (s_i) - \frac{1}{2} [x_i (s_i)]^2 + \beta x_i (s_i) \sum_{j=1}^{n} g_{ij} \mathbb{E} [x_j | s_i] \]

4.2 Equilibrium Analysis

The first-order conditions are given by
\[ \forall i \in I, \frac{\partial \mathbb{E} [u_i | s_i]}{\partial x_i} = \mathbb{E} [\alpha | s_i] - x_i^* (s_i) + \beta \sum_{j=1}^{n} g_{ij} \mathbb{E} [x_j^* | s_i] = 0 \]

Hence, the best reply of agent \( i \) is given by
\[ x_i^* (s_i) = \mathbb{E} [\alpha | s_i] + \beta \sum_{j=1}^{n} g_{ij} \mathbb{E} [x_j^* | s_i] \quad (13) \]

Using (9) and (10), we easily obtain that
\[ \widehat{\alpha}_l := \mathbb{E} [\alpha | \{s_i = l\}] = \mathbb{P} (\{\alpha = \alpha_l\} | \{s_i = l\}) \alpha_l + \mathbb{P} (\{\alpha = \alpha_h\} | \{s_i = l\}) \alpha_h \quad (14) \]
\begin{align*}
\hat{\alpha}_h := E_i [\alpha \{ s_i = h \} ] &= P(\{ \alpha = \alpha_l \} \{ s_i = h \}) \alpha_l + P(\{ \alpha = \alpha_h \} \{ s_i = h \}) \alpha_h \\
&= \frac{(1 - q)(1 - p)}{(1 - q)(1 - p) + qp} \alpha_l + \frac{qp}{(1 - q)(1 - p) + qp} \alpha_h
\end{align*}

Similarly, using (11) and (12), we get

\begin{align*}
\hat{\alpha}_h := E_i [\alpha \{ s_i = h \} ] &= P(\{ \alpha = \alpha_l \} \{ s_i = h \}) \alpha_l + P(\{ \alpha = \alpha_h \} \{ s_i = h \}) \alpha_h \\
&= \frac{(1 - q)(1 - p)}{(1 - q)(1 - p) + qp} \alpha_l + \frac{qp}{(1 - q)(1 - p) + qp} \alpha_h
\end{align*}

Denote by \( x_i := x_i(\{ s_i = l \} ) := x_i( l ) \), the action taken by agent \( i \) when receiving signal \( l \) and \( \bar{x}_i := x_i(\{ s_i = h \} ) := x_i( h ) \), the action taken by agent \( i \) when receiving signal \( h \). Using (13), for all \( i \in \mathcal{I} \), the equilibrium actions are given by:

\begin{align*}
\bar{x}_i^* &= \hat{\alpha}_l + \beta \sum_{j=1}^n g_{ij} E [x_j^* | \{ s_i = l \} ] \\
&= \hat{\alpha}_l + \beta \sum_{j=1}^n g_{ij} \left[ P(\{ s_j = l \} | \{ s_i = l \}) x_j^* + P(\{ s_j = h \} | \{ s_i = l \}) \bar{x}_j \right]
\end{align*}

and

\begin{align*}
\bar{x}_i^* &= \hat{\alpha}_l + \beta \sum_{j=1}^n g_{ij} E [x_j | \{ s_i = h \} ] \\
&= \hat{\alpha}_l + \beta \sum_{j=1}^n g_{ij} \left[ P(\{ s_j = l \} | \{ s_i = h \}) x_j^* + P(\{ s_j = h \} | \{ s_i = h \}) \bar{x}_j \right]
\end{align*}

Let us have the following notations: \( \gamma_l := P(\{ s_j = l \} | \{ s_i = l \} ) \), \( 1 - \gamma_l = P(\{ s_j = h \} | \{ s_i = l \} ) \), \( \gamma_h := P(\{ s_j = h \} | \{ s_i = h \} ) \), and \( 1 - \gamma_h = P(\{ s_j = l \} | \{ s_i = h \} ) \). Then, the optimal actions can be written as:

\begin{align*}
\bar{x}_i^* &= \hat{\alpha}_l + \beta \sum_{j=1}^n g_{ij} \left[ \gamma_l x_j^* + (1 - \gamma_l) \bar{x}_j \right] \\
&= \hat{\alpha}_l + \beta \sum_{j=1}^n g_{ij} \left[ \gamma_l x_j^* + (1 - \gamma_l) \bar{x}_j \right]
\end{align*}

We can easily calculate \( \gamma_l \) and \( \gamma_h \) as follows:
Lemma 1 We have:
\[ \gamma_l = \frac{(1 - p)q^2 + pq(1 - q)^2}{q(1 - p) + pq(1 - q)} \] (18)
and
\[ \gamma_h = \frac{(1 - p)(1 - q)^2 + pq^2}{qp + (1 - q)(1 - p)} \] (19)

Let us introduce the following notations: \( \mathbf{x} := (x_1, ..., x_n)^T \) and \( \mathbf{\bar{x}} := (\bar{x}_1, ..., \bar{x}_n)^T \) are \( n \)-dimensional vectors, \( \alpha := \left( \begin{array}{c} \hat{\alpha}_l \mathbf{1}_n \\ \hat{\alpha}_h \mathbf{1}_n \end{array} \right) \) are \( 2n \)-dimensional vectors, and
\[ \Omega := \left( \begin{array}{cc} \gamma_l \mathbf{G} & (1 - \gamma_l) \mathbf{G} \\ (1 - \gamma_h) \mathbf{G} & \gamma_h \mathbf{G} \end{array} \right) \]
is a \( 2n \times 2n \) matrix. Then the \( 2n \) equations of the best-reply functions (16) and (17) can be written in matrix form as follows:
\[ \mathbf{x}^* = \hat{\alpha} + \beta \Omega \mathbf{x}^* \]

If \( I_{2n} - \beta \Omega \) is invertible, then we obtain
\[ \left( \begin{array}{c} \mathbf{x}^* \\ \mathbf{\bar{x}}^* \end{array} \right) = \left[ I_{2n} - \beta \left( \begin{array}{cc} \gamma_l \mathbf{G} & (1 - \gamma_l) \mathbf{G} \\ (1 - \gamma_h) \mathbf{G} & \gamma_h \mathbf{G} \end{array} \right) \right]^{-1} \left( \begin{array}{c} \hat{\alpha}_l \mathbf{1}_n \\ \hat{\alpha}_h \mathbf{1}_n \end{array} \right) \] (20)
or, equivalently,
\[ \mathbf{x}^* = [I_{2n} - \beta \Omega]^{-1} \hat{\alpha} \] (21)

We can rewrite the equilibrium conditions (20) as follows:
\[ \left( \begin{array}{c} \mathbf{x}^* \\ \mathbf{\bar{x}}^* \end{array} \right) = [I_{2n} - \beta \mathbf{\Gamma} \otimes \mathbf{G}]^{-1} \left( \begin{array}{c} \hat{\alpha}_l \mathbf{1}_n \\ \hat{\alpha}_h \mathbf{1}_n \end{array} \right) \] (22)

where
\[ \mathbf{\Gamma} := \left( \begin{array}{cc} \gamma_l & 1 - \gamma_l \\ 1 - \gamma_h & \gamma_h \end{array} \right) \]
is a stochastic matrix and \( \mathbf{\Gamma} \otimes \mathbf{G} \) is the Kronecker product of \( \mathbf{\Gamma} \) and \( \mathbf{G} \) (see Appendix A.3 and, in particular, Definition 4 of the Kronecker product). \( \mathbf{\Gamma} \) is called the information matrix since it keeps track of all the information received by the agent about the states of the world while \( \mathbf{G} \), the adjacency matrix, is the “network” matrix since it keeps track of the position of each individual in the network. Our main result in this section can be stated as follows:
Proposition 2 Consider the network game with payoffs (1) and unknown parameter $\alpha$ that can only take two values: $0 < \alpha_l < \alpha_h$. Then, if $0 < \beta < 1/\lambda_{\text{max}}(G)$, there exists a unique interior Bayesian-Nash equilibrium in pure strategies given by

$$x^* = \tilde{\alpha} b(\beta, G) -\frac{(1-\gamma_l)}{(2-\gamma_h-\gamma_l)}(\tilde{\alpha}_h - \tilde{\alpha}_l) b((\gamma_h + \gamma_l - 1)\beta, G)$$

(23)

$$x^* = \tilde{\alpha} b(\beta, G) +\frac{(1-\gamma_h)}{(2-\gamma_h-\gamma_l)}(\tilde{\alpha}_h - \tilde{\alpha}_l) b((\gamma_h + \gamma_l - 1)\beta, G)$$

(24)

where

$$\tilde{\alpha} \equiv \frac{(1-\gamma_h)\hat{\alpha}_l + (1-\gamma_l)\hat{\alpha}_h}{(2-\gamma_h-\gamma_l)}$$

(25)

$\gamma_l$ and $\gamma_h$ are given by (18), and (19) and $\hat{\alpha}_l$ and $\hat{\alpha}_h$ by (14) and (15).

The following comments are in order. First, the condition for existence and uniqueness of a Bayesian-Nash equilibrium (i.e. $0 < \beta < 1/\lambda_{\text{max}}(G)$) is exactly the same as the condition for the complete information case (see Proposition 1). This is due to the fact that the information matrix $\Gamma$ is a stochastic matrix and its largest eigenvalue is thus equal to 1.

Second, we characterize the Nash equilibrium of this game for each agent and for each signal received by disentangling the network effects (captured by $G$) from the information effects (captured by $\Gamma$). We are able to do so because $G$ is symmetric and $\Gamma$ is of order 2 (i.e., it is a $2 \times 2$ matrix) and thus both are diagonalizable. We show that each effort is a combination of two Katz-Bonacich centralities where the decay factors are the eigenvalues of the information matrix $\Gamma$ times the synergy parameter $\beta$ while the weights are the conditional probabilities, which include beliefs about the states of the world given the signals received by all agents. To understand this result, observe that the diagonalization of $G$ leads to the Katz-Bonacich centrality while the diagonalization of $\Gamma$ to a matrix $A$ with eigenvectors as columns. The different eigenvalues of $\Gamma$ determine the number of the different Katz-Bonacich centrality vectors (two here) and the discount (or decay) factor in each of them $(1 \times \beta$ for the first Katz-Bonacich centrality and $(\gamma_h + \gamma_l - 1)\times \beta$ for the second Katz-Bonacich centrality, where $1$ and $\gamma_h + \gamma_l - 1$ are the two eigenvalues of $\Gamma$) and $A$ and $A^{-1}$ characterize the weights (i.e. $\hat{\alpha}$ and $\frac{(1-\gamma_h)}{(2-\gamma_h-\gamma_l)}(\hat{\alpha}_h - \hat{\alpha}_l)$) of the different Katz-Bonacich centrality vectors in equilibrium strategies.

Third, observe that $\gamma_h$ and $\gamma_l$, which are measures of the informativeness of private signals, enter both in $\Gamma$ and therefore in the Kronecker product of $\Gamma \otimes G$ and in the vector $\hat{\alpha}$. So, when $\gamma_h$ is close to 1 and $\gamma_l$ is close to 1, which means that the signals are very informative, the gap between both eigenvalues$^{12}$ (which is a measure of the entanglement of

$^{12}$The largest eigenvalue of $\Gamma$ is always 1 while the other eigenvalue is $\gamma_h + \gamma_l - 1$, so the gap between these two eigenvalues is $\gamma_h + \gamma_l - 2$.}
actions in both states) tends to 0. More generally, we should expect this to be also true in the case of $M$ different possible states of the world (we will show it formally in Section 5.3 below), bearing resemblance with the analysis in Golub and Jackson (2010, 2012), where they show that the second largest eigenvalue measures the speed of convergence of the DeGroot naive learning process, which at the same time relates to the speed of convergence of the Markov process. In our case, if the powers of $\Gamma$ stabilize very fast, we can approximate very well equilibrium actions in different states with equilibrium actions in the complete information game. Finally, note that if, for example, $\tilde{\alpha}_i - \tilde{\alpha}_h \to 0$, meaning that both levels of $\alpha$ (i.e. states of the world) are very similar, then $\mathbf{x}^* \to \tilde{\alpha}_i \mathbf{b}(\lambda; \mathbf{G})$ and $\mathbf{x}^* \to \tilde{\alpha}_h \mathbf{b}(\lambda; \mathbf{G})$. In other words, we end up with an equilibrium similar to the one obtained in the perfect information case.

5 The incomplete information case: A general model with a finite number of states and types

5.1 The model

The model of Section 4 with unknown $\alpha$ and two states of the world and two possible signals provides a good understanding on how the model works. Let us now consider a more general model when there is a finite number of states of the world and signals. We study a family of Bayesian games that share similar features and where there is incomplete information on either $\alpha$ or $\beta$. Hence we analyze Bayesian games with common values and private information (the level of direct reward of own activity, denoted by $\alpha$, and the level of pairwise strategic complementarities, denoted by $\beta$).

As above, let $\mathcal{I} := \{1, \ldots, n\}$ denote the set of players, where $n > 1$. For all $i \in \mathcal{I}$, let $s_i$ denote player $i$’s signal, where $s_i : \Omega \to S \subseteq \mathbb{R}$ is a random variable defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that $S$ is finite with $L := |\mathcal{S}| > 1$. Assume without loss of generality that $\mathcal{S} = \{1, \ldots, L\}$.$^{13}$ Let $(s_1, \ldots, s_n)^T$ denote the random $n$-vector of the players’ signals.

If $s_1, \ldots, s_n$ have the same distribution, then $s_1, \ldots, s_n$ are called identically distributed. Similarly, for all $2 \leq m \leq n$, for all $\{i_k\}_{k=1}^m \subseteq \mathcal{I}$, and for all $\{j_k\}_{k=1}^m \subseteq \mathcal{I}$, if $(s_{i_1}, \ldots, s_{i_m})^T$ and $(s_{j_1}, \ldots, s_{j_m})^T$ have the same (multivariate) distribution, then $(s_{i_1}, \ldots, s_{i_m})^T$ and $(s_{j_1}, \ldots, s_{j_m})^T$ are called identically distributed.

$^{13}$This assumption is crucial for the definition of the players’ information matrix (see Definition 3 and Remark 1).
A permutation \( \pi \) of \( \mathcal{I} \) is a bijection \( \pi : \mathcal{I} \rightarrow \mathcal{I} \). Any permutation \( \pi \) of \( \mathcal{I} \) can be uniquely represented by a non-singular \( n \times n \) matrix \( P_\pi \), the so-called permutation matrix of \( \pi \).

**Definition 2**  The (multivariate) distribution of \( (s_1, \ldots, s_n)^T \), or equivalently, the joint distribution of \( s_1, \ldots, s_n \), is called permutation invariant if for all permutations \( \pi \) of \( \mathcal{I} \),

\[
P_\pi(s_1, \ldots, s_n)^T = (s_{\pi(1)}, \ldots, s_{\pi(n)})^T \quad \text{and} \quad (s_1, \ldots, s_n)^T \quad \text{are identically distributed.}
\]

If the distribution of \( (s_1, \ldots, s_n)^T \) is permutation invariant, permuting the components of \( (s_1, \ldots, s_n)^T \) does not change its distribution. For example, if \( n = 3 \) and the (trivariate) distribution of \( (s_1, s_2, s_3)^T \) is permutation invariant, then \( (s_1, s_2, s_3)^T \), \( (s_1, s_3, s_2)^T \), \( (s_2, s_1, s_3)^T \), \( (s_2, s_3, s_1)^T \), \( (s_3, s_1, s_2)^T \), and \( (s_3, s_2, s_1)^T \) are identically distributed.

From now on, we assume that the two following assumptions hold throughout the paper:

**Assumption 1**: For all \( i \in \mathcal{I} \) and for all \( \tau \in S \), \( \mathbb{P}(\{s_i = \tau\}) > 0 \).

Assumption 1 ensures that conditional probabilities of the form \( \mathbb{P}(\{s_j = t\} \mid \{s_i = \tau\}) \) are defined.

**Assumption 2**: The distribution of \( (s_1, \ldots, s_n)^T \) is permutation invariant.

In Appendix A.1, sections A.1.1 and A.1.2, we derive some results showing the importance of each of these two assumptions. We show that the information matrix \( \Gamma \) is well-defined if Assumptions 1 and 3a (defined in Appendix A.1) are satisfied. A sufficient condition for Assumption 3a to hold true is that the distribution of the players’ signals is permutation invariant (Assumption 2). Observe that, in Proposition 5 in Appendix A.1, we show that Assumptions 1 and 2 guarantee that the eigenvalues of matrix \( \Gamma \) are all real. Observe also that Assumption 2 does not imply that \( \Gamma \) is symmetric. It just says that the identity of the player does not matter when calculating conditional probabilities. Below we give an example where Assumption 2 is satisfied and the matrix \( \Gamma \) is not symmetric.

Let us now go back to the model and let \( \theta \in \{\alpha, \beta\} \) be the unknown common value. This parameter can take \( M \) different values (i.e. states of the world), \( \theta \in \Theta = \{\theta_1, \ldots, \theta_M\} \). Agents can be of \( T \) different types, that we denote by \( S = \{1, \ldots, T\} \). These types can be interpreted as private informative signals of the value of \( \theta \). Given the type \( t \in S \) and a state \( \theta_m \in \Theta \), we denote \( p_{tm} := \mathbb{P}(\{s_i = t\} \mid \{\theta = \theta_m\}) \), \( t = 1, \ldots, T \), \( m = 1, \ldots, M \). The informational structure is therefore governed by a \( T \times M \) matrix

\[
P = (p_{tm})_{t,m} \quad t \in S, m \in \Theta,
\]
i.e.

\[
\mathbf{P} = \begin{pmatrix}
p_{11} & \cdots & p_{1M} \\
\vdots & \ddots & \vdots \\
p_{T1} & \cdots & p_{TM}
\end{pmatrix}
\]

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Next, we define the notation of the players’ information matrix.

**Definition 3** The players’ information matrix, denoted by \( \Gamma = (\gamma_{tr})_{(t,\tau)\in S^2} \), is a square matrix of order \( L = |S| \) that is given by

\[
\forall (t, \tau) \in S^2 \quad \gamma_{tr} = \mathbb{P}(\{s_i = \tau\} \mid \{s_j = t\}) = \frac{\mathbb{P}(\{s_i = \tau\} \cap \{s_j = t\})}{\mathbb{P}(\{s_j = t\})},
\]

where \((i,j) \in \mathbb{I}^2\) with \(i \neq j\) is arbitrary.

Building on this definition, we can derive the information matrix \( \Gamma = (\gamma_{tr})_{(t,\tau)\in S^2} \) where \( \gamma_{tr} \) is defined by

\[
\gamma_{tr} = \mathbb{P}(\{s_i = \tau\} \mid \{s_j = t\}) = \sum_{m=1}^{M} \frac{\mathbb{P}(\{\theta = \theta_m\} \cap \{s_i = \tau\} \mid \{s_j = t\})}{\mathbb{P}(\{s_j = t\})}
\]

i.e. \( \gamma_{tr} \) is the conditional probability of the event \( \{s_i = \tau\} \) (that is, an agent \( i \) receives the signal \( \tau \)) given the event \( \{s_j = t\} \) (that is, another agent \( j \) receives the signal \( t \)). We obtain the following \( T \times T \) matrix:

\[
\Gamma = \begin{pmatrix}
\gamma_{11} & \cdots & \gamma_{1T} \\
\vdots & \ddots & \vdots \\
\gamma_{T1} & \cdots & \gamma_{TT}
\end{pmatrix}
\]

Agents know their own type but not the type of other agents. The strategy of each agent \( i \) is the function

\[
x_i : S \longrightarrow [0, \infty)
\]

and the utility of each agent \( i \) by (1).

Let us now give an example where the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant (Assumption 2) but the matrix \( \Gamma \) is not symmetric. It readily follows from Assumption 2 that \( \mathbb{P}(\{s_i = \tau\} \cap \{s_j = t\}) = \mathbb{P}(\{s_i = t\} \cap \{s_j = \tau\}) \). This implies that the probability mass function \( \mathbb{P}(\{s_i = \tau\} \cap \{s_j = t\}) \) of the (joint) distribution of \((s_i, s_j)\) can be represented by a symmetric matrix as shown in the example below with three states of the world \( l, m \) and \( h \) and three signals:
Notice that the marginal distributions for each $s_i$ are the same. Assumption 2 therefore implies that $\mathbb{P}(\{s_i = \tau\}) = \mathbb{P}(\{s_j = \tau\})$. Indeed

$$\mathbb{P}(\{s_i = \tau\}) = \sum_{l} \mathbb{P}(\{s_i = \tau\} \cap \{s_j = t\}) = \sum_{l} \mathbb{P}(\{s_i = t\} \cap \{s_j = \tau\}) = \mathbb{P}(\{s_j = \tau\})$$

Observe, however, that this does not imply that the matrix of conditional probabilities $\Gamma$, or the information matrix, will be symmetric. Indeed, Using Definition 3, it is straightforward to derive $\Gamma$ for the above example

$$\gamma_{ll} = \mathbb{P}(\{s_i = l\}|\{s_j = l\}) = \frac{\mathbb{P}(\{s_i = l\} \cap \{s_j = l\})}{\mathbb{P}(\{s_j = l\})} = \frac{0.10}{0.25} = 0.4$$
$$\gamma_{lm} = \mathbb{P}(\{s_i = m\}|\{s_j = l\}) = \frac{\mathbb{P}(\{s_i = m\} \cap \{s_j = l\})}{\mathbb{P}(\{s_j = l\})} = \frac{0.10}{0.25} = 0.4$$
$$\gamma_{ml} = \mathbb{P}(\{s_i = l\}|\{s_j = m\}) = \frac{\mathbb{P}(\{s_i = l\} \cap \{s_j = m\})}{\mathbb{P}(\{s_j = m\})} = \frac{0.10}{0.35} = 0.286$$

It can be thus seen that $\gamma_{lr} \neq \gamma_{rl}$. Hence matrix $\Gamma$ will be non-symmetric in general. In our example, it is given by:

$$\Gamma = \begin{bmatrix} \gamma_{ll} & \gamma_{lm} & \gamma_{lh} \\ \gamma_{ml} & \gamma_{mm} & \gamma_{mh} \\ \gamma_{hl} & \gamma_{hm} & \gamma_{hh} \end{bmatrix} = \begin{bmatrix} 0.400 & 0.400 & 0.200 \\ 0.286 & 0.286 & 0.429 \\ 0.125 & 0.375 & 0.500 \end{bmatrix}$$

Observe that Assumptions 1 and 2 (see Proposition 5) guarantee that the eigenvalues are all real for matrix $\Gamma$. In the present example, it can be shown that the eigenvalues of $\Gamma$ are equal to: $\{1, 0.286, -0.1\} \subset \mathbb{R}$.

### 5.2 Examples

To illustrate our information structure, consider the following example where the number of states and types is the same, i.e., $M = T$. 

<table>
<thead>
<tr>
<th>(t/\tau)</th>
<th>(l)</th>
<th>(m)</th>
<th>(h)</th>
<th>(\mathbb{P}(s_i = \tau))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l)</td>
<td>0.10</td>
<td>0.10</td>
<td>0.05</td>
<td>0.25</td>
</tr>
<tr>
<td>(m)</td>
<td>0.10</td>
<td>0.10</td>
<td>0.15</td>
<td>0.35</td>
</tr>
<tr>
<td>(h)</td>
<td>0.05</td>
<td>0.15</td>
<td>0.20</td>
<td>0.40</td>
</tr>
<tr>
<td>(\mathbb{P}(s_j = t))</td>
<td>0.25</td>
<td>0.35</td>
<td>0.40</td>
<td>1</td>
</tr>
</tbody>
</table>

Observe that Assumptions 1 and 2 (see Proposition 5) guarantee that the eigenvalues are all real for matrix $\Gamma$. In the present example, it can be shown that the eigenvalues of $\Gamma$ are equal to: $\{1, 0.286, -0.1\} \subset \mathbb{R}$. 

### 5.2 Examples

To illustrate our information structure, consider the following example where the number of states and types is the same, i.e., $M = T$.
Example 1  Consider the case when $M = T = 2$. Denote state 1 and signal 1 by $l$, i.e., $l = 1$, and state 2 and signal 2 by $h$, i.e. $h = 2$. The prior is denoted by $p = \mathbb{P}(\alpha = \alpha_h)$ and the private signal is such that $p_l = p_{lh} = q \geq 1/2$. It follows that the $2 \times 2$ matrix $P$ is given by:

$$
P = \begin{pmatrix} q & 1 - q \\ 1 - q & q \end{pmatrix}
$$

We can then easily calculate the $2 \times 2$ matrix $\Gamma$ as follows:

$$
\Gamma = \begin{pmatrix} \gamma_{ll} & \gamma_{lh} \\ \gamma_{hl} & \gamma_{hh} \end{pmatrix}
$$

where $\gamma_{ll} := \gamma_l$, $\gamma_{lh} := 1 - \gamma_l$, $\gamma_{hh} := \gamma_h$ and $\gamma_{hl} := 1 - \gamma_h$ are given in Lemma 1.

Example 2  Let us now consider a case where there are $M = T$ states of the world (i.e., there are as many signals as possible $\theta$’s) but where the information structure is as follows. The priors are such that, for all $m \in \{1, ..., T\}$, $\mathbb{P}(\theta_m) = 1/T$. Furthermore, assume that if agent $i$ observes the signal $t$, she assigns the probability $p > 1/T$ of being in state $\theta_t$ and probability $(1 - p)/(T - 1)$ of being in each other state. In other words,

$$
p_{tm} = \mathbb{P}(\{s_i = t\} \setminus \{\theta = \theta_m\}) = \begin{cases} p & \text{if } t = m \\ (1 - p)/(T - 1) & \text{if } t \neq m \end{cases}
$$

where $p > 1/T$. In this case, the $T \times T$ matrix $P$ is given by

$$
P = \begin{pmatrix} p & \frac{1-p}{T-1} & \cdots & \frac{1-p}{T-1} \\ \frac{1-p}{T-1} & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \frac{1-p}{T-1} \\ \frac{1-p}{T-1} & \cdots & \frac{1-p}{T-1} & p \end{pmatrix} = p\mathbf{1}_n + \left(\frac{1-p}{T-1}\right) (\mathbf{1}_n\mathbf{1}_n^T - \mathbf{I}_n) \quad (27)
$$

where $\mathbf{1}_n$ is a $n$–dimensional vector of ones and $\mathbf{I}_n$ is the $n$–dimensional identity matrix. It is easily verified that $P$ is symmetric. Let us now determine the $T \times T$ information matrix $\Gamma$. First, let us compute the values of $\mathbb{P}(\{\theta = \theta_m\} \cap \{s_j(\tau)|s_i(t)\})$ for given $t, \tau, m$. We have that

$$
\mathbb{P}(\{\theta = \theta_m\} \cap \{s_j = \tau\}) \{s_i = t\}) = \frac{\mathbb{P}(\{\theta = \theta_m\} \cap \{s_j = \tau\} \cap \{s_i = t\})}{\mathbb{P}(\{s_i = t\})} \\
= \frac{\mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\} \cap \{\theta = \theta_m\})\mathbb{P}(\{\theta = \theta_m\})}{\mathbb{P}(\{s_i = t\})} \\
= \frac{\mathbb{P}(\{\theta = \theta_m\} \cap \{s_j = \tau\} \cap \{s_i = t\})\mathbb{P}(\{s_i = t\})}{\mathbb{P}(\{s_i = t\})}
$$

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Given the assumptions on the informational structure, we have that \( \mathbb{P}(\theta_m) = 1/T \) and
\[
\mathbb{P}\left(\{s_i = t\}\right) = \sum_{m=1}^{n} \mathbb{P}(\{s_i = t\} \mid \{\theta = \theta_m\}) \mathbb{P}(\{\theta = \theta_m\}) = p \times \frac{1}{T} + \left(\frac{1-p}{T-1}\right)(T-1) \times \frac{1}{T} = \frac{1}{T}
\]

Thus the signal \( s_i \) of player \( i \) has the discrete uniform distribution on \( S = \{1, ... , T\} \) (Assumption 3). As a result,
\[
\mathbb{P}(\{\theta = \theta_m\} \cap \{s_j = \tau\} \mid \{s_i = t\}) = \mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\} \mid \{\theta = \theta_m\}) = \mathbb{P}(\{\theta = \theta_m\} \cap \{s_j = \tau\} \mid s_i(t))
\]

It is then easy to show that:
\[
\mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\} \mid \{\theta = \theta_m\}) = \begin{cases} 
(\frac{1-p}{T-1})^2 & \text{if } \tau \neq m \text{ and } t \neq m \\
p(\frac{1-p}{T-1}) & \text{if either } (\tau \neq m \text{ and } t = m) \text{ or } (\tau = m \text{ and } t \neq m) \\
p^2 & \text{if } \tau = t = m
\end{cases}
\]

We have:
\[
\gamma_{tr} = \mathbb{P}(\{s_j = \tau\} \mid \{s_i = t\}) = \sum_{m=1}^{M} \mathbb{P}(\{\theta = \theta_m\} \cap \{s_j = \tau\} \mid \{s_i = t\}) = \sum_{m=1}^{M} \mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\} \mid \{\theta = \theta_m\})
\]

Hence, if \( t = \tau \), we obtain:
\[
\gamma_{tr} = p^2 + (T-1) \left(\frac{1-p}{T-1}\right)^2 = p^2 + \frac{(1-p)^2}{T-1}
\]

while, if \( t \neq \tau \), we get:
\[
\gamma_{tr} = 2p \left(\frac{1-p}{T-1}\right) + (T-2) \left(\frac{1-p}{T-1}\right)^2
\]
\[
= \frac{(1-p)(Tp + T - 2)}{(T-1)^2}
\]
The information matrix is thus given by:

\[
\Gamma = \begin{pmatrix}
p^2 + \frac{(1-p)^2}{T-1} & \frac{(1-p)(T_p+T-2)}{(T-1)^2} & \cdots & \frac{(1-p)(T_p+T-2)}{(T-1)^2} \\
\frac{(1-p)(T_p+T-2)}{(T-1)^2} & p^2 + \frac{(1-p)^2}{T-1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \frac{(1-p)(T_p+T-2)}{(T-1)^2} \\
\frac{(1-p)(T_p+T-2)}{(T-1)^2} & \cdots & \frac{(1-p)(T_p+T-2)}{(T-1)^2} & p^2 + \frac{(1-p)^2}{T-1}
\end{pmatrix}
\]  

Equation (28)

Evidently, \( \Gamma \) is symmetric.

### 5.3 The model with unknown \( \alpha \)

Let us now solve the model with unknown \( \alpha \) when there is a finite number of states of the world \( (M) \) and signals \( (T) \).

#### 5.3.1 Equilibrium

Assume that the \( T \times T \) information matrix \( \Gamma \) is diagonalizable. Then,

\[
\Gamma = AD_T A^{-1}
\]

where

\[
D_T = \begin{pmatrix}
\lambda_1(\Gamma) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_T(\Gamma)
\end{pmatrix}
\]

and \( \lambda_1(\Gamma), \ldots, \lambda_T(\Gamma) \) are the eigenvalues of \( \Gamma \), with \( \lambda_{\max}(\Gamma) := \lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \ldots \geq \lambda_T(\Gamma) \).

In this formulation, \( A \) is a \( T \times T \) matrix where each \( i \)th column is formed by the eigenvector corresponding to the \( i \)th eigenvalue. Let us have the following notations:

\[
A = \begin{pmatrix}
 a_{i1} & \cdots & a_{iT} \\
\vdots & \ddots & \vdots \\
a_{T1} & \cdots & a_{TT}
\end{pmatrix}
\]  

and

\[
A^{-1} = \begin{pmatrix}
 a_{11}^{(-1)} & \cdots & a_{1T}^{(-1)} \\
\vdots & \ddots & \vdots \\
a_{T1}^{(-1)} & \cdots & a_{TT}^{(-1)}
\end{pmatrix}
\]

where \( a_{ij}^{(-1)} \) is the \( (i, j) \) cell of the matrix \( A^{-1} \).
The utility function of individual receiving signal $\tau$ can be written as:

$$
E[u_i \{ s_i = \tau \}] = E[\alpha \{ s_i = \tau \}] x_i(\tau) - \frac{1}{2} [x_i(\tau)]^2 + \beta x_i(\tau) \sum_{j=1}^{n} g_{ij} E[x_j \{ s_i = \tau \}]
$$

The first order conditions are given by

$$
\frac{\partial E[u_i \{ s_i = \tau \}]}{\partial x_i} = E[\alpha \{ s_i = \tau \}] - x_i^*(\tau) + \beta \sum_{j=1}^{n} g_{ij} E[x_j^* \{ s_i = \tau \}]
$$

$$
= E[\alpha \{ s_i = \tau \}] - x_i^*(\tau) + \beta \sum_{j=1}^{n} \sum_{t=1}^{T} g_{ij} \mathbb{P}\{ \{ s_j = t \} | \{ s_i = \tau \} \} x_j^*(\tau)
$$

$$
= E[\alpha \{ s_i = \tau \}] - x_i^*(\tau) + \beta \sum_{j=1}^{n} \sum_{t=1}^{T} g_{ij} \gamma_{jt} x_j^*(\tau)
$$

Define

$$
\forall \tau \in \{1,...,T\} \quad \hat{\alpha}_t := E[\alpha \{ s_i = \tau \}] = \sum_{m=1}^{M} \alpha_m \mathbb{P}(\{ \alpha = \alpha_m \} | \{ s_i = \tau \})
$$

We have the following result.

**Theorem 1** Consider the case when the marginal return of effort is unknown. Assume that $\Gamma$ is diagonalizable. Let $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq ... \geq \lambda_T(\Gamma)$ be the eigenvalues of the information matrix $\Gamma$. Then, if $\{ \alpha_t \}_{t=1}^{T} \subset \mathbb{R}_{++}$ and $0 < \beta < 1/\lambda_{\max}(G)$, there exists a unique Bayesian-Nash equilibrium. In that case, if the signal received is $s = \tau$, then the equilibrium efforts are given by:

$$
x^*\{s = \tau\} = \hat{\alpha}_1 \sum_{t=1}^{T} a_{t\tau} a_{t1}^{-1} b(\lambda_{t}\Gamma_{t} \beta, G) + ... + \hat{\alpha}_T \sum_{t=1}^{T} a_{t\tau} a_{tT}^{-1} b(\lambda_{t}\Gamma_{t} \beta, G) \quad (29)
$$

for $\tau = 1,...,T$.

Theorem 1 generalizes Proposition 2 when there are $M$ states of the world, $\theta \in \Theta = \{ \theta_1, \ldots, \theta_M \}$, and $T$ different signals or types, $S = \{1, ..., T\}$. Interestingly, the condition for existence and uniqueness of a Bayesian-Nash equilibrium (i.e. $0 < \beta < 1/\lambda_{\max}(G)$) is still the same because $\Gamma$ is still a stochastic matrix whose largest eigenvalue is 1. The proof of this Theorem is relatively similar to that of Proposition 2 where we diagonalize the two matrices $G$ and $\Gamma$ to obtain a nice characterization of the equilibrium conditions. The characterization obtained in Theorem 1 is such that each equilibrium effort (or action) is a
combination of the $T$ different Katz-Bonacich centralities, where the decay factors are the corresponding eigenvalues of the information matrix $\Gamma$ multiplied by the synergy parameter $\beta$, while the weights are the elements of the $A$ and $A^{-1}$. This is because the diagonalization of $G$ leads to the Katz-Bonacich centralities while the diagonalization of $\Gamma$ leads to a matrix $A$, with eigenvectors as columns. This implies that the number of the different eigenvalues of $\Gamma$ determines the number of the different Katz-Bonacich centrality vectors and the discount factor in each of them, while the elements of $A$ and $A^{-1}$ characterize the weights of the different Katz-Bonacich vectors in equilibrium strategies.

Observe that, in Theorem 1, we assume that $\Gamma$ is diagonalizable. The case of non-diagonalizable $\Gamma$ is nongeneric (Meyer, 2001). However, this does not mean that such matrices could not occur in practice. We consider the case of a non-diagonalizable $\Gamma$ in Section 6 below.

In the next section, we give some examples showing how this characterization is easy to implement.

5.3.2 Examples

Example 1 Consider the example of the previous section where $\alpha$ takes only two values: $\alpha_l$ and $\alpha_h$, with $\alpha_l < \alpha_h$ (i.e. $M = T = 2$). In this case, we have:

$$\Gamma = \begin{pmatrix} \gamma_l & 1 - \gamma_l \\ 1 - \gamma_h & \gamma_h \end{pmatrix}$$

where $\lambda_1(\Gamma) = \lambda_{\max}(\Gamma) = 1$ and $\lambda_2(\Gamma) = \gamma_h + \gamma_l - 1$ and

$$A = \begin{pmatrix} 1 & 1 - (1 - \gamma_l) / (1 - \gamma_h) \\ 1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 - \gamma_h / (2 - \gamma_l - \gamma_h) & (1 - \gamma_l) / (1 - \gamma_h) \\ -1 & 1 \end{pmatrix}$$

Let us calculate the equilibrium conditions given in (29), where the state $l$ corresponds to 1 and the state $h$ corresponds to 2 ($m = l, h; t = l, h$ and $M = T = 2$). Using (29), if agent $i$ receives the signal $s_i = l$, we easily obtain:

$$x^*_i(l) := \hat{x}^*_i = \alpha_l \left[ a_{ll} a_{l}^{(-1)} b_l (\lambda_1(\Gamma) \beta, G) + a_{lh} a_{h}^{(-1)} b_l (\lambda_2(\Gamma) \beta, G) \right] + \alpha_h \left[ a_{hl} a_{h}^{(-1)} b_l (\lambda_1(\Gamma) \beta, G) + a_{hh} a_{h}^{(-1)} b_l (\lambda_2(\Gamma) \beta, G) \right]$$

Since $a_{ll} = a_{hh} = a_{hl} = 1$, $a_{lh} = -\left(1 - \gamma_l \right) \left(1 - \gamma_h \right)$, $a_{ll}^{(-1)} = a_{lh}^{(-1)} = \frac{1 - \gamma_l}{2 - \gamma_l - \gamma_h}$, $a_{hh}^{(-1)} = -\left(1 - \gamma_h \right)$, $a_{lh}^{(-1)} = \frac{1 - \gamma_l}{2 - \gamma_l - \gamma_h}$, $\lambda_1(\Gamma) = 1$ and $\lambda_2(\Gamma) = \gamma_h + \gamma_l - 1$, it is easily verified that $x^*_i(l)$ is exactly
equal to (23). Using (29), with the same methodology, it is easy to show that $x^*_t (h) := \pi^*_t$ is equal to (24).

If we assume, for example, that $p = 0.6$ and $q = 0.85$, then, using Lemma 1, we have: $\gamma_l = 0.703$ and $\gamma_h = 0.776$ and thus $\lambda_2 (\Gamma) = 0.48$. Assume also that $\beta = 0.2$, which means that $\lambda_1 (\Gamma) \beta = 0.2$ and $\lambda_2 (\Gamma) \beta = 0.096$, then

$$\begin{align*}
x^* &= \hat{\alpha} b (0.2, G) - 0.57 (\hat{\alpha}_h - \hat{\alpha}_l) b (0.096, G) \\
x^* &= \hat{\alpha} b (0.2, G) + 0.43 (\hat{\alpha}_h - \hat{\alpha}_l) b (0.096, G)
\end{align*}$$

(30)

If we further assume that $\alpha_l = 0.2$ and $\alpha_h = 0.8$, then: $\hat{\alpha}_l = 0.326$, $\hat{\alpha}_h = 0.737$ and $\hat{\alpha} = 0.56$. Thus,

$$\begin{align*}
x^* &= 0.56 b (0.2, G) - 0.234 b (0.096, G) \\
x^* &= 0.56 b (0.2, G) + 0.177 b (0.096, G)
\end{align*}$$

(31)

To understand the importance of networks, consider the star network of Section 3.2 (Figure 1). Then, the condition $\beta < 1/\lambda_{\text{max}} (G^S)$ can be written as $0.2 < 1/\sqrt{2} = 0.707$, which is always satisfied. As a result, the Katz-Bonacich centralities are given by:

$$\begin{align*}
\begin{pmatrix}
b_1 (0.2, G^S) \\
b_2 (0.2, G^S) \\
b_3 (0.2, G^S)
\end{pmatrix}
= \begin{pmatrix}
1.522 \\
1.304 \\
1.304
\end{pmatrix}
\text{ and }
\begin{pmatrix}
b_1 (0.093, G^S) \\
b_2 (0.093, G^S) \\
b_3 (0.093, G^S)
\end{pmatrix}
= \begin{pmatrix}
1.214 \\
1.117 \\
1.117
\end{pmatrix}
\end{align*}$$

In that case, there exists a unique Nash equilibrium, which is given by:

$$\begin{align*}
\begin{pmatrix}
\pi_1^{S*} = 0.568 \\
\pi_2^{S*} = 0.469 \\
\pi_3^{S*} = 0.469
\end{pmatrix}
\text{ and }
\begin{pmatrix}
\pi_1^{S*} = 1.067 \\
\pi_2^{S*} = 0.928 \\
\pi_3^{S*} = 0.928
\end{pmatrix}
\end{align*}$$

Consider now the complete network with the same three agents so that:14

$$G^C = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}$$

The condition $\beta < 1/\lambda_{\text{max}} (G^C)$ can now be written as $0.2 < 1/2 = 0.5$, which is satisfied. Then,

$$\begin{align*}
\begin{pmatrix}
b_1 (0.2, G^C) \\
b_2 (0.2, G^C) \\
b_3 (0.2, G^C)
\end{pmatrix}
= \begin{pmatrix}
1.667 \\
1.667 \\
1.667
\end{pmatrix}
\text{ and }
\begin{pmatrix}
b_1 (0.096, G^C) \\
b_2 (0.096, G^C) \\
b_3 (0.096, G^C)
\end{pmatrix}
= \begin{pmatrix}
1.238 \\
1.238 \\
1.238
\end{pmatrix}
\end{align*}$$

14The superscript $C$ refers to the complete network.
In that case,
\[
\begin{pmatrix}
\ell_1^{C^*} = 0.643 \\
\ell_2^{C^*} = 0.643 \\
\ell_3^{C^*} = 0.643
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\ell_1^{C^*} = 1.152 \\
\ell_2^{C^*} = 1.152 \\
\ell_3^{C^*} = 1.152
\end{pmatrix}
\]
Not surprisingly, the efforts are always higher in the complete network compared to the star network due to more strategic complementarities. However, the high-to-low-effort ratio is lower in the complete network and less central agents have a higher ratio. Indeed,
\[
\begin{pmatrix}
\ell_1^{C^*} = 1.000 \\
\ell_2^{C^*} = 1.000 \\
\ell_3^{C^*} = 1.000
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\ell_1^{C^*} = 1.791 \\
\ell_2^{C^*} = 1.791 \\
\ell_3^{C^*} = 1.791
\end{pmatrix}
\]
Finally, we can compare our results with the complete information case where optimal efforts are given by (6). Assume as above that \(\beta = 0.2\) and \(\alpha = (\alpha_l + \alpha_h) / 2 = 0.5\). In that case, \(\ell_1^{S^*} = 0.761\) and \(\ell_2^{S^*} = \ell_3^{S^*} = 0.652\). For the complete network, we have: \(\ell_1^{C^*} = \ell_2^{C^*} = \ell_3^{C^*} = 0.833\). If we now take the average value of effort in the incomplete information case (i.e. \((\ell_1 + \ell_1^*) / 2)\), we obtain:
\[
\begin{pmatrix}
\ell_1^{S^*} = 0.817 \\
\ell_2^{S^*} = 0.698 \\
\ell_3^{S^*} = 0.698
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\ell_1^{C^*} = 0.898 \\
\ell_2^{C^*} = 0.898 \\
\ell_3^{C^*} = 0.898
\end{pmatrix}
\]
It is interesting to see that, on average, individuals put more effort under incomplete information.

**Example 2** Let us now consider the model of Section 5.2 with \(M = T\) and where the \(T \times T\) information matrix \(\Gamma\) is given by (28). Assume that \(p = 0.6\) and \(T = 3\). This means that \(\alpha\) can take three values \(\alpha_l, \alpha_w, \alpha_h\) and that each agent receives a signal, which is either equal to \(l, w\) or \(h\). In that case,
\[
\Gamma = \begin{pmatrix}
0.44 & 0.28 & 0.28 \\
0.28 & 0.44 & 0.28 \\
0.28 & 0.28 & 0.44
\end{pmatrix}
\quad (32)
\]
This matrix \(\Gamma\) has two distinct eigenvalues: \(\lambda_1(\Gamma) = 1\) and \(\lambda_2(\Gamma) = 0.16\). We can thus diagonalize \(\Gamma\) as follows:
\[
\Gamma = A \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.16 & 0 \\
0 & 0 & 0.16
\end{pmatrix} A^{-1}
\]
\[24\]
where
\[
\begin{bmatrix}
0.577 & -0.765 & 0.286 \\
0.577 & 0.630 & 0.520 \\
0.577 & 0.135 & -0.805
\end{bmatrix}, \quad
\begin{bmatrix}
0.577 & 0.577 & 0.577 \\
-0.765 & 0.630 & 0.135 \\
0.286 & 0.520 & -0.805
\end{bmatrix}
\]
(33)

Assuming as before that \(\beta = 0.2\), which means that \(\lambda_1(\Gamma)\beta = 0.2\) and \(\lambda_2(\Gamma)\beta = 0.032\). Therefore, applying Theorem 1, if each agent \(i\) receives the signal \(s_i = l\), then her equilibrium effort is equal to:

\[
x^*(l) = \alpha_l [0.333 b(0.2, G) + 0.667 b(0.032, G)] \\
+ \alpha_w [0.333 b(0.2, G) - 0.333 b(0.032, G)] \\
+ \alpha_h [0.333 b(0.2, G) - 0.333 b(0.032, G)]
\]

Similar calculations can be done when each agent \(i\) receives the signals \(s_i = w\) and \(s_i = h\).

In Appendix A.2, we derive the same results for the case when \(\beta\) is unknown. In particular, Theorem 4 gives the conditions for existence and uniqueness of a Bayesian-Nash equilibrium and its characterization. We also provide some examples showing how this characterization can be implemented.

### 6 Diagonalizable versus nondiagonalizable information matrix \(\Gamma\)

In Theorem 1, we assumed that the information matrix \(\Gamma\) was diagonalizable, which is generically true. First, let us give some sufficient condition on the primitives of the model (i.e. on the joint distribution of the signals) that guarantees that \(\Gamma\) is symmetric and thus diagonalizable. We have the following assumption:

**Assumption 3:** The signal \(s_i\) of each player \(i\) has the discrete uniform distribution on \(S\).

In Appendix A.1, Section A.1.3, we show in Proposition 6 that, if Assumption 3 holds, then the information matrix \(\Gamma\) is symmetric, and therefore diagonalizable. In some sense, Assumption 3 is relatively similar to what is assumed in the literature with linear-quadratic utility functions and a continuum of signals (and states of the world) where the signals are assumed to follow a Normal distribution (see e.g. Calvó-Armengol and de Martí, 2009, and Bergemann and Morris, 2013).\(^{15}\) Note that there are other, potentially less restrictive

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\(^{15}\)Indeed, the uniform distribution defined on a finite set is the maximum entropy distribution among all discrete distributions supported on this set. Similarly, the Normal distribution \(N(\mu, \sigma^2)\) has maximum entropy among all real-valued distributions with specified mean \(\mu\) and standard deviation \(\sigma\).
conditions that would ensure the diagonalizability of \( \Gamma \). For example, it is sufficient to assume that \( \Gamma \) is strictly sign-regular. Then, it can be shown that all its eigenvalues will be real, distinct, and thus simple and the corresponding eigenbasis will consist of real vectors (see Ando, 1987, Theorem 6.2). The advantage of Assumption 3 is that it provides some sufficient conditions in terms of more primitive assumptions on the joint distribution of the signals.

Second, when \( \Gamma \) is nondiagonalizable, we can still characterize our unique Bayesian-Nash equilibrium using the Jordan decomposition. We have the following result, whose proof is given in Appendix A.4:

**Theorem 2** Consider the case when the marginal return of effort is unknown. Let \( \lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \ldots \geq \lambda_T(\Gamma) \) be the eigenvalues of the information matrix \( \Gamma \). Assume that the Jordan form of \( \Gamma \) is made up of \( Q \) Jordan blocks \( J(\tilde{\lambda}_q) \), where \( \tilde{\lambda}_q \) is the eigenvalue associated with the \( q \)-th Jordan block of \( \Gamma \). Let \( \tilde{\lambda}_1(\Gamma) \geq \tilde{\lambda}_2(\Gamma) \geq \ldots \geq \tilde{\lambda}_Q(\Gamma) \). Let \( d_q \) be the dimension of the \( q \)-th Jordan block of \( \Gamma \), and define \( D_q := \sum_{q=1}^Q d_q \), and let \( u_{n,h}(\tilde{\lambda}_q) := (I_n - \tilde{\lambda}_q \beta G)^{-k} \beta^k G^k 1_n \). Then, if \( \{\tilde{\alpha}_\tau\}_{\tau=1}^T \subset \mathbb{R}_{++} \) and \( 0 < \beta < 1/\lambda_{\max}(G) \), there exists a unique Bayesian-Nash equilibrium. In that case, if the signal received is \( s = \tau \), then the equilibrium efforts are given by:

\[
x^*\{s = \tau\} = \tilde{\alpha}_1 \sum_{q=1}^Q D_{q-1} + d_q \sum_{h=1}^{D_q} a_{\tau h} d_{h1}^{(-1)} b_{u_{n-h}}(\tilde{\lambda}_q \beta, G) + \ldots + \tilde{\alpha}_T \sum_{q=1}^Q D_{q-1} + d_q \sum_{h=1}^{D_q} a_{\tau h} d_{hT}^{(-1)} b_{u_{n-h}}(\tilde{\lambda}_q \beta, G)
\]

for \( \tau = 1, \ldots, T \), or, more compactly

\[
x^*\{s = \tau\} = \sum_{t=1}^T \tilde{\alpha}_t \sum_{q=1}^Q D_{q-1} + d_q \sum_{h=1}^{D_q} a_{\tau h} d_{h1}^{(-1)} b_{u_{n-h}}(\tilde{\lambda}_q \beta, G)
\]

where \( b_{u_{n-h}}(\tilde{\lambda}_q \beta, G) \) denotes the \( u_{n-h} \)-weighted Katz-Bonacich centrality.

We can see that the structure of the equilibrium characterization (34) is similar to that of Theorem 1, given by (29). It contains, however, additional terms, which are weighted Katz-Bonacich centralities \( b_{u_{n-h}}(\tilde{\lambda}_q \beta, G) \), and is more complicated to calculate. The main advantage of this result is that it does not hinge on the diagonalizability of the information matrix \( \Gamma \). Observe that the number and the weights of the Katz-Bonacich centralities given
in (34) depend on the deficiency of the information matrix $\Gamma$. This implies that, when $\Gamma$ is diagonalizable so that its eigenvalues are either simple or semi-simple, then the equilibrium characterization of efforts given by (34) collapses to (29), which is given by Theorem 1.

7 Key-player policies

We would like now to derive some policy implications of our model. In the context of the model above with complete information, Ballester et al. (2006, 2010) have studied the “key player” policy, which consists of finding the key player, i.e. the individual who, once removed from the network, generates the highest possible reduction in aggregate activity. This is particularly relevant for crime (Liu et al., 2012; Lindquist and Zenou, 2014) but also for financial networks (Denbee et al., 2014), R&D networks (König et al., 2014), and development economics (Banerjee et al., 2013).16 In Section 7.1, we will first expose this policy in the context of complete information. Then, in Section 7.2, we will analyze this policy when information is incomplete.

7.1 Perfect information: Key players

7.1.1 Model

Consider the utility function defined by (1). At the Nash equilibrium, each individual $i$ will provide effort given by (6), i.e. $x^*_i = \alpha b_i (\beta, G)$ (see Proposition 1). Denote by

$$x^*(G) = \sum_{i=1}^{n} x^*_i = \alpha \sum_{i=1}^{n} b_i (\beta, G)$$

the total sum of efforts at the Nash equilibrium. The planner’s objective is to generate the highest possible reduction in aggregate effort level by picking the appropriate individual. Formally, the planner’s problem is the following:

$$\max_{i \in \{1, \ldots, n\}} \{x^*(G) - x^*(G^{-i})\}$$

where $G^{-i}$ is the $(n - 1) \times (n - 1)$ adjacency matrix corresponding to the network $G^{-i}$ when individual $i$ has been removed. From Ballester et al. (2006, 2010), we now define a new network centrality measure, called the intercentrality of agent $i$, and denoted by $IC_i (\beta, G)$ that will solve this program:

\footnote{For an overview of the literature on key players in networks, see Zenou (2015b).}

27
Proposition 3 Assume $\alpha > 0$ and $0 < \beta < 1/\lambda_{\text{max}}(G)$. Then, under complete information, the key player in network $g$ is the agent $i$\textsuperscript{17} that has the highest intercentrality measure $IC_i(\beta, G)$, which is defined by:

$$IC_i(\beta, G) := \frac{[b_i(\beta, G)]^2}{m_{ii}(\beta, G)}$$

(35)

where $m_{ii}(\beta, G)$ is the cell corresponding to the $i$th row and the $i$th column of the matrix $(I_n - \beta G)^{-1}$ and thus keeps track of the paths that start and finish at $i$ (cycles).

This proposition says that the key player $i^*$ who solves $\max_{i \in \{1, \ldots, n\}}\{x^*(G) - x^*(G^{-i})\}$ is the agent who has the highest inter-centrality $IC_i(\beta, G)$ in $g$, that is, for all $i = 1, \ldots, n$, $IC_{i^*}(\beta, G) \geq IC_i(\beta, G)$. As a result,

$$IC_{i^*}(\beta, G) \in \max_{i \in \{1, \ldots, n\}}\{x^*(G) - x^*(G^{-i})\}$$

(36)

7.1.2 Example 3

Consider the bridge network $g^B$ in Figure 2 with eleven agents.

![Bridge network](image)

Figure 2: Bridge network

We distinguish three different types of equivalent actors in this network, which are the following:

<table>
<thead>
<tr>
<th>Type</th>
<th>Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2, 6, 7, and 11</td>
</tr>
<tr>
<td>3</td>
<td>3, 4, 5, 8, 9, and 10</td>
</tr>
</tbody>
</table>

Table 1 computes, for agents of types 1, 2 and 3 the value of the Katz-Bonacich centrality measures $b_i(\beta, G^B)$ (which is equal to the effort $x_i^*$ when $\alpha = 1$) and the intercentrality

\textsuperscript{17}The key player might not be unique.
measures $IC_i(\beta, G^B)$ for different values of $\beta$. In each column, a variable with a star identifies the highest value.\footnote{We can compute the highest possible value for $\beta$ compatible with our definition of centrality measures. It is equal to $$\beta = \frac{1}{\lambda_{\text{max}}(G)} = \frac{1}{4.404} = 0.227$$}

<table>
<thead>
<tr>
<th>Type</th>
<th>$b_i(0.096, G^B)$</th>
<th>$IC_i(0.096, G^B)$</th>
<th>$b_i(0.2, G^B)$</th>
<th>$IC_i(0.2, G^B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.679</td>
<td>2.726</td>
<td>7.143</td>
<td>35.714*</td>
</tr>
<tr>
<td>2</td>
<td>1.793*</td>
<td>3.016*</td>
<td>7.381*</td>
<td>28.601</td>
</tr>
<tr>
<td>3</td>
<td>1.646</td>
<td>2.570</td>
<td>6.191</td>
<td>21.949</td>
</tr>
</tbody>
</table>

Table 1: Katz-Bonacich versus intercentrality measures in a bridge network

First note that type–2 agents always display the highest Katz-Bonacich centrality measure. These agents have the highest number of direct connections. Besides, they are directly connected to the bridge agent 1, which gives them access to a very wide and diversified span of indirect connections. For low values of $\beta$, the direct effect on effort reduction prevails, and type–2 agents are the key players —those with highest intercentrality measure $IC_i(\beta, G^B)$. When $\beta$ is higher, though, the most active agents are not anymore the key players. Now, indirect effects matter a lot, and eliminating agent 1 has the highest joint direct and indirect effect on aggregate effort reduction.

### 7.2 Incomplete information on $\alpha$: Key players

#### 7.2.1 Model with two states of the world

As in Section 4, we assume that there are two states of the world,\footnote{The general case with $M$ states of the world is considered below.} so that the parameter $\alpha$ can take two values: $\alpha_l < \alpha_h$. We also assume that the planner has a prior (which is unknown to the agents) for the event $\{\alpha = \alpha_h\}$, which is given by:

$$P(\{\alpha = \alpha_h\}) = p^A \in (0, 1)$$

The prior of the planner may be different than the one shared by the agents, which is $P(\{\alpha = \alpha_h\}) = p \in (0, 1)$ because the planner may have superior information. The planner
needs to solve the key player problem, which is the difference in aggregate activity according to her prior, i.e. \( \max_{i \in \{1, \ldots, n\}} \Delta X_i \), where

\[
\Delta X_i = \left[ p^A \mathbf{\bar{x}}^*(G) + (1 - p^A) \mathbf{\bar{x}}^*(G^{-i}) \right] - \left[ p^A \mathbf{\bar{x}}^*(G^{-i}) + (1 - p^A) \mathbf{\bar{x}}^*(G^{-i}) \right]
\]

where \( \mathbf{\bar{x}}^* \) are the Bayesian-Nash equilibrium high actions defined by (24) while \( \mathbf{\bar{x}}^* \) are the Bayesian-Nash equilibrium low actions defined by (23), and \( \mathbf{\bar{x}}^*(G) = \sum_{j=1}^n \mathbf{x}^*_j, \mathbf{\bar{x}}^*(G^{-i}) = \sum_{j=1, j \neq i}^n \mathbf{x}^*_j \). Indeed, if the planner believes that the state of the world is \( \mathcal{A}_h \), which occurs with probability \( \pi_\mathcal{A}_h \), then she believes that all agents will play the high actions while, if it is \( \mathcal{A}_l \), then she thinks that the low actions will be played.

Using the values defined in (24) and (23), we obtain:

\[
\Delta X_i = \hat{\alpha} \left[ \sum_{j=1}^n b_j (\beta, G) - \sum_{j=1}^{n-1} b_j (\beta, G^{-i}) \right] \\
+ \omega \left[ \sum_{j=1}^n b_j ((\gamma_h + \gamma_l - 1) \beta, G) - \sum_{j=1}^{n-1} b_j ((\gamma_h + \gamma_l - 1) \beta, G^{-i}) \right]
\]

where

\[
\omega := (\hat{\alpha}_h - \hat{\alpha}_l) \left[ p^A \frac{(1 - \gamma_h)}{(2 - \gamma_h - \gamma_l)} - (1 - p^A) \frac{(1 - \gamma_l)}{(2 - \gamma_h - \gamma_l)} \right]
\]

and \( \hat{\alpha} \) is defined in (25), \( \gamma_l \) and \( \gamma_h \) are given by (18) and (19), and \( \hat{\alpha}_l \) and \( \hat{\alpha}_h \) by (14) and (15). Using the definition of intercentrality given in (36), this is equivalent to

\[
\Delta X_i = \hat{\alpha} IC_i (\beta, G) + \omega IC_i ((\gamma_h + \gamma_l - 1) \beta, G)
\]

We have the following proposition:

**Proposition 4** Assume \( \{\hat{\alpha}_x\}_{x=1}^2 \subset \mathbb{R}_{++} \) and \( 0 < \beta < 1/\lambda_{\max}(G) \). Then, under incomplete information on \( \alpha \) and with two states of the world, the key player is given by:

\[
\arg \max_{i \in \{1, \ldots, n\}} \{ \Delta IC_i (\beta, G, \Gamma) \equiv \hat{\alpha} IC_i (\beta, G) + \omega IC_i ((\gamma_h + \gamma_l - 1) \beta, G) \}
\]

If \( \omega = 0 \), then, when \( \hat{\alpha}_l - \hat{\alpha}_h \to 0 \), which means that both levels of \( \alpha \) (i.e. state of the world) are very similar, the optimal targeting is equivalent to the complete information case. If \( \omega \neq 0 \), then the optimal targeting may change. This is because the ranking derived from the intercentrality measure is not stable over \( \beta \).
7.2.2 A general model with a finite states of the world

It is relatively easy to extend the model when there is a finite number of states of the world and the uncertainty is on $\alpha$. In that case, the planner has expectations $p_m^A$ when the state of the world is $m$, with $\sum_{m=1}^{M} p_m^A = 1$. For simplicity and to avoid too cumbersome notations, assume that when the planner believes that the state is $\alpha_m$ (with probability $p_m^A$), then each agent $i$ receives the signal $s_i(\{\tau = m\})$ so that $T = M$. Using (29), the expected aggregate activity when the network is $G$ is given by:

$$\sum_{m=1}^{M} p_m^A \sum_{j=1}^{n} x_j(s(m), G)$$

$$= \sum_{m=1}^{T} p_m^A \sum_{j=1}^{n} \left( \sum_{t=1}^{T} \tilde{a}_t a_{mt} a_{t1} (-1) b_j (\lambda_t (\Gamma) \beta, G) + \ldots + \sum_{t=1}^{T} \tilde{a}_t a_{mt} a_{tT} (-1) b_j (\lambda_t (\Gamma) \beta, G) \right)$$

$$= \sum_{m=1}^{M} p_m^A \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \omega_{\tau mt} b_j (\lambda_t (\Gamma) \beta, G)$$

where $\omega_{\tau mt} := \tilde{a}_t a_{mt} a_{t\tau} (-1)$. The expected aggregate activity when the network is $G^{-i}$ can be written exactly in the same way by replacing $G$ with $G^{-i}$. The key player is thus defined as:

$$\arg \max_{i \in N} \left\{ \sum_{m=1}^{M} p_m^A \sum_{j=1}^{n} x_j(s(m), G) - \sum_{m=1}^{M} p_m^A \sum_{j=1}^{n-1} x_j(s(m), G^{-i}) \right\}$$

$$= \arg \max_{i \in N} \left\{ \sum_{m=1}^{M} p_m^A \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \omega_{\tau mt} b_j (\lambda_t (\Gamma) \beta, G) - \sum_{m=1}^{M} p_m^A \sum_{j=1}^{n-1} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \omega_{\tau mt} b_j (\lambda_t (\Gamma) \beta, G^{-i}) \right\}$$

Given the definition of intercentrality in (36), we have the following result:

**Theorem 3** Assume that $\{\tilde{\alpha}_\tau\}_{\tau=1}^{T} \subset \mathbb{R}_{++}$ and $0 < \beta < 1/\lambda_{\max}(G)$. Then, under incomplete information with $T$ states of the world, the key player $i^* \in N$ is the agent that solves

$$\arg \max_{i \in N} \left\{ \Delta IC_i (\beta, G, \Gamma) \equiv \sum_{m=1}^{M} p_m^A \sum_{t=1}^{T} \sum_{\tau=1}^{T} \omega_{\tau mt} IC_i (\lambda_t (\Gamma) \beta, G) \right\}$$

It is straightforward to determine the key player when the uncertainty is on $\beta$ and the results are very similar to the case when $\alpha$ is unknown. We will now provide an example showing how the key player is determined when there is incomplete information on $\alpha$. 31
7.2.3 Examples

Example 1  Let us go back to Example 1 (Section 5.3.2). Let us first calculate the Bayesian-Nash equilibrium when $\alpha$ can take two values $\alpha_l < \alpha_h$ using Proposition 2. Assume as above that $p = 0.6$ and $q = 0.85$, so that $\gamma_l = 0.703$ and $\gamma_h = 0.776$. Assume also that $\beta = 0.2$. We have seen (see (30)) that

$$x^* = \hat{\alpha} b(0.2, G) - 0.57(\hat{\alpha}_h - \hat{\alpha}_l) b(0.096, G)$$
$$x^- = \hat{\alpha} b(0.2, G) + 0.43(\hat{\alpha}_h - \hat{\alpha}_l) b(0.096, G)$$

Let us now calculate the key player for the bridge network $g^B$ with eleven agents displayed in Figure 2 (Example 3 in Section 7.1.2). The key player is the agent $i$ that maximizes

$$\Delta IC_i(\beta, G^B, \Gamma) \equiv \hat{\alpha} IC_i(0.2; G) + \omega IC_i(0.096; G)$$

Using Table 1, we obtain:

<table>
<thead>
<tr>
<th>Type</th>
<th>$b_i(0.096, G^B)$</th>
<th>$IC_i(0.096, G^B)$</th>
<th>$b_i(0.2, G^B)$</th>
<th>$IC_i(0.2, G^B)$</th>
<th>$\hat{\alpha} IC_i(0.2) + \omega IC_i(0.096)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.679</td>
<td>2.726</td>
<td>7.143</td>
<td>35.714*</td>
<td>35.714 $\hat{\alpha}$ + 2.726 $\omega$</td>
</tr>
<tr>
<td>2</td>
<td>1.793*</td>
<td>3.016*</td>
<td>7.381*</td>
<td>28.601</td>
<td>28.601 $\hat{\alpha}$ + 3.016 $\omega$</td>
</tr>
<tr>
<td>3</td>
<td>1.646</td>
<td>2.570</td>
<td>6.191</td>
<td>21.949</td>
<td>21.949 $\hat{\alpha}$ + 2.57 $\omega$</td>
</tr>
</tbody>
</table>

Table 2: Intercentrality measures in a bridge network with imperfect information

Hence, the key player depends on the ratio $\omega/\hat{\alpha}$, which is given by:

$$\frac{\omega}{\hat{\alpha}} = \frac{(\hat{\alpha}_h - \hat{\alpha}_l) \left[p^A(1 - \gamma_h) - (1 - p^A)(1 - \gamma_l)\right]}{(1 - \gamma_h) \hat{\alpha}_l + (1 - \gamma_l) \hat{\alpha}_h}$$

This ratio captures all the (incomplete) information that the agents have: the priors $p$ and $p^A$ of the agents and the planner and the posteriors of the agents displayed in the elements of the matrices $P$ and $\Gamma$. If the ratio $\omega/\hat{\alpha}$ is small, type–1 agents are more likely to be the key players as in the perfect information case. However, if $\omega/\hat{\alpha}$ is high, then type–2 agents will be the key players, which give the opposite prediction compared to the perfect information case. It readily verified that $\omega/\hat{\alpha}$ increases with $p^A$, $\gamma_h$, and $\hat{\alpha}_h$ and decreases with $\gamma_h$ and $\hat{\alpha}_l$. To understand this result, two effects need to be considered: (i) there is a
common effect when \(|\omega|\) is large (e.g. high variance of \(\hat{\alpha}\)); (ii) there is an idiosyncratic effect when \(\gamma_H + \gamma_L - 1 \ll 1\) and then the change in ranking is more likely to occur.

To illustrate this result, assume, as before, that \(\alpha_l = 0.2\), \(\alpha_l = 0.8\), so that \(\hat{\alpha}_l = 0.326\), \(\hat{\alpha}_l = 0.737\) and \(\hat{\alpha} = 0.56\). Also assume that \(p = 0.6\) and \(q = 0.85\), so that \(\gamma_l = 0.703\) and \(\gamma_h = 0.776\). Finally, assume \(p^A = 0.8 > p\). Then using (37), we have: \(\omega = 0.0945\), which means that \(\omega/\hat{\alpha} = 0.169\). It is easily verified that, when \(\alpha\) is (partially) unknown, the key player is individual 1 since she is the one who has the highest \(\hat{\alpha}IC_1(0.2) + \omega IC_1(0.096)\), while, when there is perfect information on \(\alpha\), the key player is individual 2 if \(\beta\) is low enough (for example, if \(\beta = 0.096\); see Table 2) and individual 1 if \(\beta\) is high enough (for example, if \(\beta = 0.2\); see Table 2). If we now change the parameters of the model so that \(\omega/\hat{\alpha}\) becomes larger (by assuming higher values of \(p^A\), \(\gamma_h\), and \(\hat{\alpha}_h\) and lower values of \(\gamma_h\) and \(\hat{\alpha}_l\)), then the key player in the imperfect information case becomes individual 2.

**Example 2** Let us now consider the model of Section 5.2 with \(M = T\) and where the \(T \times T\) information matrix \(\Gamma\) is given by (28). Assume that \(p = 0.6\) and \(T = 3\). This means that \(\alpha\) can take three values, \(\alpha_l\), \(\alpha_w\), \(\alpha_h\), and that each agent \(i\) receives of the different three signals: \(l\), \(w\) or \(h\). State 1 corresponds to \(l\), state 2 to \(w\) and state 3 to \(h\). In that case, \(\Gamma\) is given by (32) and \(A\) and \(A^{-1}\) by (33). Since the stochastic matrix \(\Gamma\) has two distinct eigenvalues: \(\lambda_1(\Gamma) = 1\) and \(\lambda_2(\Gamma) = 0.16\). Assume as before that \(\beta = 0.2\), which means that \(\lambda_1(\Gamma) \beta = 0.2\) and \(\lambda_2(\Gamma) \beta = 0.032\). Then, we easily obtain:

<table>
<thead>
<tr>
<th>Type</th>
<th>(b_1(0.032, G^B))</th>
<th>(IC_1(0.032, G^B))</th>
<th>(b_1(0.2, G^B))</th>
<th>(IC_1(0.2, G^B))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.151</td>
<td>1.319</td>
<td>7.143</td>
<td>35.714*</td>
</tr>
<tr>
<td>2</td>
<td>1.184*</td>
<td>1.394*</td>
<td>7.381*</td>
<td>28.601</td>
</tr>
<tr>
<td>3</td>
<td>1.148</td>
<td>1.312</td>
<td>6.191</td>
<td>21.949</td>
</tr>
</tbody>
</table>

Table 3: Katz-Bonacich versus intercentrality measures in a bridge network

We obtain exactly the same result as in Example 1, i.e., the key player changes depending on the value of \(\beta\) so that, when \(\beta\) is small, the key player is individual 2 while, it is individual 1 when \(\beta\) is higher. The intercentrality measure in the imperfect information case is now
The calculation of $\Delta IC_i (\beta, G^B, \Gamma)$ is much more complicated but it is still a weighted average of intercentrality measures and the weights are the different parameters related to the information structure since $a_{mt}a_{tt}^{(-1)}$ are elements of $A$ and $A^{-1}$, $\hat{\alpha}_t = \sum_{m=1}^M P(\alpha_m|s_i(t)) \alpha_m$ are conditional expectations on the $\alpha$s and $p_m^A$ are the prior of the planner. As for example 1, the same tradeoff arises between the $\omega$s and the $\hat{\alpha}$s.

8 Conclusion

We analyze a family of tractable network games with incomplete information on relevant payoff parameters. We show under which condition there exists a unique Bayesian-Nash equilibrium. We are also able to explicitly characterize this Bayesian-Nash equilibrium by showing how it depends in a very precise way on both the network geometry (Katz-Bonacich centrality) and the informational structure. Finally, we prove that incomplete information can distort the network policy implications with respect to the complete information benchmark. Indeed, the targeting of individuals in a network (key-player policy) may be very different when information of the payoff structure is partially known.

We believe that, in many situations, information is incomplete and thus our model can shed light on these issues. For example, in criminal networks (Ballester et al., 2010; Calvó-Armengol and Zenou, 2004; Liu et al., 2013), delinquents do not always know the proceeds from crime (unknown $\alpha$ in our model) or the synergy their obtain from other delinquents (unknown $\beta$). In R&D networks (Goyal and Moraga-Gonzalez, 2001; König et al., 2014), the marginal reduction in production costs due to R&D collaboration is partially known. In education (Calvó-Armengol et al., 2009), all students know each other connections in the classroom, i.e. the network structure, but they may not be completely aware of what are the benefits of studying. In financial networks (Acemoglu et al., 2015; Cohen-Cole et al., 2011; Denbee et al., 2014; Elliott et al., 2014), this is even more true. Banks do not know exactly the risk taken by giving loans to other banks. From an empirical perspective, it would interesting to bring the model to the data. If we know the network (as, for example, in the AddHealth data; see e.g. Calvó-Armengol et al., 2009), we can estimate the quality of
information of a group and how it impacts the outcomes of agents. We leave this for future research.

References


A Appendix

A.1 Main assumptions of the model and their implications

A.1.1 Some useful results

Lemma 2 If the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant, then \(s_1, \ldots, s_n\) are identically distributed.

Proof. Assume that the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant. Let \((i, j) \in \mathcal{I}^2 \text{ with } i \neq j\), and let \(\pi\) be a permutation of \(\mathcal{I}\) with \(\pi(i) = j\). Let \(B_i \subset \mathbb{R}\) be a Borel set, for example, \(B_i = \{\tau\}\) for some \(\tau \in \mathcal{S}\), and for all \(k \in \mathcal{I} \setminus \{i\}\), let \(B_k = \mathbb{R}\). We find

\[
\mathbb{P}(\{s_i \in B_i\}) = \mathbb{P}\left(\bigcap_{k=1}^{n}\{s_k \in B_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{n}\{s_{\pi(k)} \in B_k\}\right) = \mathbb{P}(\{s_j \in B_i\}).
\]

The first equality follows from the fact that

\[
\{s_i \in B_i\} = \{s_i \in B_i\} \cap \Omega = \{s_i \in B_i\} \cap \bigcap_{k=1,k \neq i}^{n}\{s_k \in B_k\} = \bigcap_{k=1}^{n}\{s_k \in B_k\}.
\]

The second equality follows from the assumption that the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant. The third equality follows from the fact that

\[
\bigcap_{k=1}^{n}\{s_{\pi(k)} \in B_k\} = \{s_{\pi(i)} \in B_i\} \cap \bigcap_{k=1,k \neq i}^{n}\{s_{\pi(k)} \in B_k\}
\]

\[
= \{s_{\pi(i)} \in B_i\} \cap \Omega = \{s_{\pi(i)} \in B_i\} = \{s_j \in B_i\}.
\]

We conclude that \(s_1, \ldots, s_n\) are identically distributed. \(\blacksquare\)

Lemma 3 If the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant, then for all \((i, j) \in \mathcal{I}^2 \text{ with } i \neq j\) and for all \((k, l) \in \mathcal{I}^2 \text{ with } k \neq l\), \((s_i, s_j)^T\) and \((s_k, s_l)^T\) are identically distributed.

Proof. Assume that the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant. Let \((i, j) \in \mathcal{I}^2 \text{ with } i \neq j\), and let \((k, l) \in \mathcal{I}^2 \text{ with } k \neq l\). Let \(\pi\) be a permutation of \(\mathcal{I}\) with \(\pi(i) = k\) and
Lemma 4

for all pairs of signal values is (functionally) independent of \( I \) because Assumption 3a is satisfied. Let \( \{\tau\} \) for some \( \tau \in S \) and \( \{t\} \) for some \( t \in S \), and for all \( k \in T \setminus \{i, j\} \), let \( B_k = \mathbb{R} \). We find

\[
P((s_i, s_j)^T \in B_i \times B_j) = P(\{s_i \in B_i \} \cap \{s_j \in B_j \})
\]

\[
= P\left( \bigcap_{k=1}^{n} \{s_k \in B_k \} \right)
\]

\[
= P\left( \bigcap_{k=1}^{n} \{s_{\pi(k)} \in B_k \} \right)
\]

\[
= P(\{s_{\pi(i)} \in B_i \} \cap \{s_{\pi(j)} \in B_j \})
\]

\[
= P(\{s_k \in B_i \} \cap \{s_l \in B_j \})
\]

\[
= P((s_k, s_l)^T \in B_i \times B_j).
\]

The third equality follows from the assumption that the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant. The other equalities are obvious. We conclude that \((s_i, s_j)^T\) and \((s_k, s_l)^T\) are identically distributed. ■

A.1.2 Main results using Assumptions 1 and 2

We introduce the following assumption concerning the distribution of the players’ signals.

Assumption 3a: For all \((i, j) \in T^2\) with \(i \neq j\), for all \((k, l) \in T^2\) with \(k \neq l\), and for all \((t, \tau) \in S^2\), \(P(\{s_k = \tau\})P(\{s_j = t\} \cap \{s_i = \tau\}) = P(\{s_i = \tau\})P(\{s_l = t\} \cap \{s_k = \tau\})\).

Suppose Assumption 1 (defined in the text) is satisfied. Then, Assumption 3a states that for all pairs of signal values \((t, \tau) \in S^2\), the conditional probability \(P(\{s_j = t\} \mid \{s_i = \tau\})\) is (functionally) independent of \((i, j) \in T^2\) or, equivalently, \(P(\{s_j = t\} \mid \{s_i = \tau\})\) is only a function of \((t, \tau)\) but not of \((i, j)\), where \(i \neq j\).

The following lemma gives a sufficient condition for Assumption 3a to be satisfied.

Lemma 4 If the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant (Assumption 2), then Assumption 3a is satisfied.

Proof. Assume that the distribution of \((s_1, \ldots, s_n)^T\) is permutation invariant. Let \((i, j) \in T^2\) with \(i \neq j\), \((k, l) \in T^2\) with \(k \neq l\), and \((t, \tau) \in S^2\). We find

\[
P(\{s_k = \tau\})P(\{s_j = t\} \cap \{s_i = \tau\}) = P(\{s_i = \tau\})P(\{s_l = t\} \cap \{s_k = \tau\})
\]

because \(P(\{s_k = \tau\}) = P(\{s_i = \tau\})\) according to Lemma 2 and \(P(\{s_j = t\} \cap \{s_i = \tau\}) = P(\{s_l = t\} \cap \{s_k = \tau\})\) according to Lemma 3. ■
**Remark 1** Note that $S = \{1, \ldots, L\}$. If $S \neq \{1, \ldots, L\}$, then we could not directly use $S$ as an index set to define the components of $\Gamma$. Clearly, $\Gamma$ can still be defined in a reasonable way if $S \neq \{1, \ldots, L\}$. To see this, suppose $S \neq \{1, \ldots, L\}$. There exists a unique order isomorphism $h: S \to \{1, \ldots, L\}$. Using $h$, we can restate Definition 3 as follows: Suppose Assumptions 1 and 3a are satisfied. The players’ information matrix, denoted by $\Gamma = (\gamma_{rs})_{(r,s) \in \{1, \ldots, L\}^2}$, is a square matrix of order $L = |S|$ that is given by

$$
\gamma_{rs} = \mathbb{P}(\{s_j = h^{-1}(s)\} \cap \{s_i = h^{-1}(r)\}) \quad \mathbb{P}(\{s_i = h^{-1}(r)\}),
$$

where $(i, j) \in I^2$ with $i \neq j$ is arbitrary.

We conclude this Appendix with a statement about the spectrum of $\Gamma$.

**Proposition 5** Suppose Assumptions 1 and 2 are satisfied. Then the eigenvalues of $\Gamma$ are real.

**Proof.** Suppose Assumption 1 is satisfied and assume that the distribution of $(s_1, \ldots, s_n)^T$ is permutation invariant. Let $(i, j) \in I^2$ with $i \neq j$. Let $\Lambda = (\lambda_{r,t})_{(r,t) \in S^2}$ be the diagonal matrix of order $L$ given by

$$
\lambda_{r,t} = \mathbb{P}(\{s_i = \tau\}) = \mathbb{P}^i(\{s_i = \tau\}),
$$

where $\mathbb{P}^i(\{s_i = \tau\})$ is the probability of the event that $s_i$ takes the value $\tau$. We write $\Lambda^{-1} = (\lambda_{r,t}^{-1})_{(r,t) \in S^2}$. Let $\Sigma = (\sigma_{r,t})_{(r,t) \in S^2}$ be the square matrix of order $L$ given by

$$
\sigma_{r,t} = \mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\}).
$$

Note that $\Sigma$ is symmetric because the distribution of $(s_1, \ldots, s_n)^T$ is permutation invariant. We have $\Lambda^{-1} \Sigma = \Gamma$. Indeed, for all $(r, t) \in S^2$,

$$\sum_{k=1}^{L} \lambda_{r,k}^{-1} \sigma_{k,t} = \lambda_{r,t}^{-1} \sigma_{r,t}
= \frac{1}{\mathbb{P}(\{s_i = \tau\})} \mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\})
= \mathbb{P}(\{s_j = t\} \mid \{s_i = \tau\})
= \gamma_{r,t}.
$$

\[20\] An order isomorphism is an order-preserving bijection.
Since \( \Lambda \) is symmetric and positive definite, it has a unique square root \( \Lambda^{1/2} \), which is symmetric and positive definite (and therefore non-singular). Let \( \Lambda^{-1/2} \) denote the inverse of \( \Lambda^{1/2} \). We have

\[
\Lambda^{1/2} \Lambda^{-1/2} = \Lambda^{1/2} (\Lambda^{-1} \Sigma) \Lambda^{-1/2} = \Lambda^{-1/2} \Sigma \Lambda^{-1/2},
\]

that is, \( \Gamma \) is similar to the symmetric matrix \( \Lambda^{-1/2} \Sigma \Lambda^{-1/2} \). Note that the spectrum of \( \Lambda^{-1/2} \Sigma \Lambda^{-1/2} \) is real because it is symmetric. We conclude that the spectrum of \( \Gamma \) is real because similar matrices have the same spectrum.

A.1.3 Main result using Assumption 3

**Proposition 6** If the distribution of \( (s_1, \ldots, s_n)^T \) is permutation invariant and \( s_1 \) has the discrete uniform distribution on \( S \), then \( \Gamma^T = \Gamma \), that is, \( \Gamma \) is symmetric.

**Proof.** Assume that the distribution of \( (s_1, \ldots, s_n)^T \) is permutation invariant and \( s_1 \) has the discrete uniform distribution on \( S \). It follows that \( s_1, \ldots, s_n \) are identically distributed with the discrete uniform distribution on \( S \) (Lemma 2). Let \( (t, \tau) \in S^2 \). We need to show that \( \gamma_{\tau,t} = \gamma_{t,\tau} \). We find

\[
\gamma_{\tau,t} = \frac{\mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\})}{\mathbb{P}(\{s_i = \tau\})} = \frac{\mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\})}{\mathbb{P}(\{s_i = t\})} = \gamma_{t,\tau}.
\]

The first and the third equality are according to (26). The second equality follows from the assumption that the distribution of \( (s_1, \ldots, s_n)^T \) is permutation invariant and \( s_1 \) has the discrete uniform distribution on \( S \). Indeed, \( \mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\}) = \mathbb{P}(\{s_i = t\} \cap \{s_j = \tau\}) = \mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\}) \) because the \( (s_j, s_i)^T \) and \( (s_i, s_j)^T \) are identically distributed (Lemma 3) and \( \mathbb{P}(\{s_i = \tau\}) = \mathbb{P}(\{s_i = t\}) \) because \( s_i \) has the discrete uniform distribution of \( S \).

A.2 The model with unknown \( \beta \)

Assume that the uncertainty is on the synergy parameter \( \beta \), which is an unknown common value for all agents. As in the case of unknown \( \alpha \), there are \( M \) different states of the world so that \( \beta \) can take \( M \) different values: \( \beta \in \{\beta_1, \ldots, \beta_M\} \). There are \( T \) different values for a signal, so that agents can be of \( T \) different types, which we denote by \( S = \{1, \ldots, T\} \).
A.2.1 Equilibrium

When agent \( i \) receives the signal \( s_i = \tau \), individual \( i \) computes the following conditional expected utility:

\[
E[ u_i | \{ s_i = \tau \} ] = \alpha E[ x_i | \{ s_i = \tau \} ] - \frac{1}{2} E[ x_i^2 | \{ s_i = \tau \} ] + \sum_{j=1}^{n} g_{ij} x_i E[ \beta x_j | \{ s_i = \tau \} ]
\]

\[
= \alpha x_i (\tau) - \frac{1}{2} x_i (\tau)^2 + \sum_{j=1}^{n} g_{ij} x_i (\tau) E[ \beta x_j | \{ s_i = \tau \} ]
\]

The first-order conditions are given by:

\[
\forall i = 1, \ldots, n \quad \frac{\partial E[ u_i | \{ s_i = \tau \} ]}{\partial x_i (\tau)} = \alpha - x_i^*(\tau) + \sum_{j=1}^{n} g_{ij} E[ \beta x_j^* | \{ s_i = \tau \} ] = 0
\]

When agent \( i \) receives the signal \( s_i = \tau \), for each possible \( j \), we have that

\[
E[ \beta x_j | \{ s_i = \tau \} ] = \sum_{t=1}^{T} \sum_{m=1}^{M} \beta_m x_j(t) P(\{ \beta = \beta_m \} \cap \{ s_j = t \} | \{ s_i = \tau \})
\]

\[
= \sum_{t=1}^{T} \sum_{m=1}^{M} P(\{ \beta = \beta_m \} \cap \{ s_j = t \} | \{ s_i = \tau \}) \beta_m x_j(t)
\]

\[
= \sum_{t=1}^{T} \left( \sum_{m=1}^{M} P(\{ \beta = \beta_m \} \cap \{ s_j = t \} | \{ s_i = \tau \}) \beta_m \right) x_j(t)
\]

\[
= \beta_{\text{max}} \sum_{t=1}^{T} \left( \sum_{m=1}^{M} P(\{ \beta = \beta_m \} \cap \{ s_j = t \} | \{ s_i = \tau \}) \beta_m \right) x_j(t)
\]

where \( \beta_{\text{max}} := \max \{ \beta_1, \ldots, \beta_M \} \). We define a \( T \times T \) matrix \( \widehat{\Gamma} \) with entries equal to \( (\widehat{\gamma}_{\tau t})_{(\tau, t) \in \{1, \ldots, T\}^2} \) where \( \widehat{\gamma}_{\tau t} \) is defined by (individual \( i \) receives signal \( \tau \) while individual \( j \) receives signal \( t \)):

\[
\widehat{\gamma}_{\tau t} = \sum_{m=1}^{M} P(\{ \beta = \beta_m \} \cap \{ s_j = t \} | \{ s_i = \tau \}) \frac{\beta_m}{\beta_{\text{max}}}
\]
Observe that, in the case of incomplete information on $\beta$ instead of $\alpha$, for all $m \in \{1, \ldots, M\}$, $\frac{\beta_m}{\beta_{\max}} \leq 1$. Therefore, $\overline{\Gamma}$ is non-stochastic because

$$
\sum_{t=1}^{T} \widetilde{\gamma}_{rt} = \sum_{t=1}^{T} \sum_{m=1}^{M} \mathbb{P}(\{\beta = \beta_m\} \cap \{s_j = t\} \mid \{s_i = \tau\}) \frac{\beta_m}{\beta_{\max}} < \sum_{t=1}^{T} \sum_{m=1}^{M} \mathbb{P}(\{\beta = \beta_m\} \cap \{s_j = t\} \mid \{s_i = \tau\}) = 1
$$

Altogether, this means that we can write the first order conditions as follows:

$$
\alpha - \alpha^{*}_{t}(\tau) + \beta_{\max} \sum_{j=1}^{n} a_{ij} \sum_{t=1}^{T} \widetilde{\gamma}_{rt} x_j(t) = 0
$$

Therefore the system of the best-replies is now given by:

$$
\begin{pmatrix}
    x(1) \\
    \vdots \\
    x(T)
\end{pmatrix} = \left( I_{Tn} - \beta_{\max} \frac{\Gamma}{\text{information}} \otimes \frac{\mathbf{G}}{\text{network}} \right)^{-1} \begin{pmatrix}
    \alpha 1 \\
    \vdots \\
    \alpha 1
\end{pmatrix}
$$

To characterize the equilibrium, we can use the same techniques as for the case when $\alpha$ was unknown. We have the following result.

**Theorem 4** Consider the case when the strength of interactions is unknown. Assume that $\overline{\Gamma}$ is is diagonalizable. Let $\lambda_{1}(\overline{\Gamma}) \geq \cdots \geq \lambda_{T}(\overline{\Gamma})$ be the eigenvalues of the information matrix $\overline{\Gamma}$ and $\lambda_{1}(\mathbf{G}) \geq \cdots \geq \lambda_{n}(\mathbf{G})$ be the eigenvalues of the adjacency matrix $\mathbf{G}$ where $\lambda_{\max}(\overline{\Gamma}) = \max_{t} \left\{ \lambda_{t}(\overline{\Gamma}) \right\}$ and $\lambda_{\max}(\mathbf{G}) = \max_{t} \{ \left| \lambda_{t}(\mathbf{G}) \right| \}$. Then, there exists a unique Bayesian-Nash equilibrium if $\alpha > 0$, $\beta_{\min} > 0$ and

$$
\beta_{\max} < \frac{1}{\lambda_{\max}(\overline{\Gamma}) \lambda_{\max}(\mathbf{G})}
$$

If the signal received is $s_i = \tau$, the equilibrium efforts are given by:

$$
\mathbf{x}^{*}(\tau) = \alpha \left[ \sum_{t=1}^{T} a_{\tau t} a_{t1}^{-1} \mathbf{b} \left( \lambda_{t}(\overline{\Gamma}) \beta_{\max} ; \mathbf{G} \right) \ldots + \sum_{t=1}^{T} a_{\tau t} a_{t1}^{-1} \mathbf{b} \left( \lambda_{t}(\overline{\Gamma}) \beta_{\max} ; \mathbf{G} \right) \right]
$$

for $j = 1, \ldots, T$. 

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Proof: See Appendix A.5.

The results are relatively similar to the case when \( \alpha \) was unknown. One of the main difference with Theorem 1 is that the condition (39) is weaker since it imposes a larger upper bound on \( \beta_{\text{max}} \) compared to \( \beta_{\text{max}} < 1/\lambda_{\text{max}}(G) \) because \( \Gamma \) is not stochastic and its largest eigenvalue \( \lambda_{\text{max}}(\Gamma) \) is not 1. We have the following remark that shows, however, that the condition \( \beta_{\text{max}} < 1/\lambda_{\text{max}}(G) \) is still a sufficient condition.

Remark 2 A sufficient condition for existence and uniqueness of a Bayesian-Nash equilibrium when \( \beta \) is unknown is that \( \beta_{\text{max}} < 1/\lambda_{\text{max}}(G) \).

Proof: See Appendix A.5.

Theorem 4 gives a complete characterization of equilibrium efforts as a function of weighted Katz-Bonacich centralities when \( \beta \) is unknown. In the next section, we show how to calculate these equilibrium efforts for some specific networks and specific information structures.

A.2.2 Examples

Example 1 Consider first a model which is equivalent to that of Section 4, where state \( l \) corresponds to 1 and state \( h \) corresponds to 2 \((m = l, h; t = l, h \) and \( M = T = 2 \)). Here, \( \beta \) takes two values: \( \beta_l \) and \( \beta_h \), with \( \beta_l < \beta_h \). In that case, we have:

\[
\Gamma = \begin{pmatrix} \gamma_l & 1 - \gamma_l \\ 1 - \gamma_h & \gamma_h \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} \gamma_l \frac{\beta_l}{\beta_h} & (1 - \gamma_l) \frac{\beta_l}{\beta_h} \\ 1 - \gamma_h & \gamma_h \end{pmatrix}
\]

The two eigenvalues can be calculated but have cumbersome values. To simplify, assume as in Section 5.3.2 that \( p = 0.6 \) and that \( q = 0.85 \) so that \( \gamma_l = 0.703 \) and \( \gamma_h = 0.776 \). Also, assume that \( \beta_l = 0.2 \) and \( \beta_h = \beta_{\text{max}} = 0.4 \). Then

\[
\Gamma = \begin{pmatrix} 0.352 & 0.149 \\ 0.224 & 0.776 \end{pmatrix}
\]

and the two eigenvalues are: \( \lambda_1(\Gamma) = 0.844 \) and \( \lambda_2(\Gamma) = 0.284 \). Then:

\[
A = \begin{pmatrix} 0.290 & 0.910 \\ 0.957 & -0.414 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0.418 & 0.918 \\ 0.966 & -0.293 \end{pmatrix}
\]
We can now use Theorem 4 and state that, if \( \lambda_{\text{max}}(G) < 0.4 \times 0.844 = 2.962 \), then if individual \( i \) receives the signal \( s_i = l \), she provides a unique effort \( x_i^*(\{s_i = l\}) := \mathcal{L}_i^* \) given by:

\[
\mathcal{L}_i^* = \alpha [a_{il}a_{il}^{(-1)}b_i (0.338, G) + a_{lh}a_{ih}^{(-1)}b_i (0.114, G) + a_{lh}a_{ih}^{(-1)}b_i (0.338, G) + a_{lh}a_{ih}^{(-1)}b_i (0.114, G)] \\
= \alpha [0.121 b_i (0.338, G) + 0.879 b_i (0.114, G) + 0.266 b_i (0.338, G) - 0.267 b_i (0.114, G)]
\]

If individual \( i \) receives the signal, \( s_i = h \), she provides a unique effort \( x_i^*(\{s_i = h\}) := \mathcal{H}_i^* \), which is given by:

\[
\mathcal{H}_i^* = \alpha [a_{hi}a_{lh}^{(-1)}b_i (0.338, G) + a_{hh}a_{ih}^{(-1)}b_i (0.114, G) + a_{hh}a_{ih}^{(-1)}b_i (0.338, G) + a_{hh}a_{ih}^{(-1)}b_i (0.114, G)] \\
= \alpha [0.4 b_i (0.338, G) - 0.4 b_i (0.114, G) + 0.879 b_i (0.338, G) + 0.121 b_i (0.114, G)]
\]

Let us consider the star network of Figure 1 (Section 3.2), then \( \lambda_{\text{max}}(G^S) = \sqrt{2} < 2.962 \). We have

\[
\begin{pmatrix}
    b_1 (0.338, G^S) \\
    b_2 (0.338, G^S) \\
    b_3 (0.338, G^S)
\end{pmatrix} = \begin{pmatrix} 2.172 \\ 1.734 \\ 1.734 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 (0.114, G^S) \\ b_2 (0.114, G^S) \\ b_3 (0.114, G^S) \end{pmatrix} = \begin{pmatrix} 1.261 \\ 1.144 \\ 1.144 \end{pmatrix}
\]

In that case, if \( \alpha = 0.5 \), we have:

\[
\begin{pmatrix}
    \mathcal{L}_1^{S*} = 0.806 \\
    \mathcal{L}_2^{S*} = 0.686 \\
    \mathcal{L}_3^{S*} = 0.686
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
    \mathcal{H}_1^{S*} = 1.213 \\
    \mathcal{H}_2^{S*} = 0.949 \\
    \mathcal{H}_3^{S*} = 0.949
\end{pmatrix}
\]

If we now consider the complete network with three agents (where \( \lambda_{\text{max}}(G^C) = 2 < 2.962 \), obtain:

\[
\begin{pmatrix}
    b_1 (0.338, G^C) \\
    b_2 (0.338, G^C) \\
    b_3 (0.338, G^C)
\end{pmatrix} = \begin{pmatrix} 3.086 \\ 3.086 \\ 3.086 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 (0.114, G^C) \\ b_2 (0.114, G^C) \\ b_3 (0.114, G^C) \end{pmatrix} = \begin{pmatrix} 1.295 \\ 1.295 \\ 1.295 \end{pmatrix}
\]

and thus

\[
\begin{pmatrix}
    \mathcal{L}_1^{C*} = 0.993 \\
    \mathcal{L}_2^{C*} = 0.993 \\
    \mathcal{L}_3^{C*} = 0.993
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
    \mathcal{H}_1^{C*} = 1.793 \\
    \mathcal{H}_2^{C*} = 1.793 \\
    \mathcal{H}_3^{C*} = 1.793
\end{pmatrix}
\]
Example 2 Let us now consider the model of Section 5.2 with $\mathbb{P} = T$ and where the $T \times T$ matrix $P$ is given by (27) and the $T \times T$ information matrix $\Gamma$ by (28). We want to compute the matrix $\tilde{\Gamma}$ and then get a closed-form expression for the Bayesian-Nash equilibrium. We have seen that

$$
\mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\} \mid \{\theta = \theta_m\})
= \begin{cases} 
(\frac{1-p}{T-1})^2 & \text{if } \tau \neq m \text{ and } t \neq m \\
p \left(\frac{1-p}{T-1}\right) & \text{if either } (\tau \neq m \text{ and } t = m) \text{ or } (\tau = m \text{ and } t \neq m) \\
p^2 & \text{if } \tau = t = m
\end{cases}
$$

and that, if $t = \tau$, we obtain:

$$
\gamma_{\tau \tau} = p^2 + \frac{(1-p)^2}{T-1}
$$

while, if $t \neq \tau$, we get:

$$
\gamma_{\tau t} = \frac{(1-p)(Tp + T - 2)}{(T - 1)^2}
$$

It is easily verified that, if $t = \tau$, then

$$
\tilde{\gamma}_{\tau \tau} = \frac{1}{\beta_{\text{max}}} \left( p^2 \beta_\tau + \left( \frac{1-p}{T-1} \right)^2 \sum_{m \neq \tau} \beta_m \right)
= \frac{1}{\beta_{\text{max}}} \left[ \left( p^2 - \left( \frac{1-p}{T-1} \right)^2 \right) \beta_\tau + \left( \frac{1-p}{T-1} \right)^2 T \hat{\beta} \right]
$$

where $\hat{\beta}$ is the expected value of $\beta$ with respect to the prior distribution, i.e. $\hat{\beta} = \frac{1}{\tau} \sum_m \beta_m = \frac{1}{\tau} \sum_m \beta_m$. If, on the contrary, $t \neq \tau$, then

$$
\tilde{\gamma}_{\tau t} = \sum_m \mathbb{P}(\beta_m, s(t) \mid s(\tau)) \frac{\beta_m}{\beta_{\text{max}}} = \sum_m \mathbb{P}(s(t), s(\tau) \mid \beta_m) \frac{\beta_m}{\beta_{\text{max}}}
= \frac{1}{\beta_{\text{max}}} \left( p \left( \frac{1-p}{T-1} \right) (\beta_\tau + \beta_t) + \left( \frac{1-p}{T-1} \right)^2 \sum_{m \neq \tau} \beta_m \right)
= \frac{1}{\beta_{\text{max}}} \left( \frac{1-p}{T-1} \right) \left[ \left( \frac{Tp - 1}{T - 1} \right) (\beta_\tau + \beta_t) + \left( 1 - \frac{Tp - 1}{T - 1} \right) \hat{\beta} \right]
$$

Clearly the matrix $\tilde{\Gamma}$ is symmetric and thus diagonalizable.
As above, assume that $p = 0.6$ and $T = 3$. This means that $\beta$ can take three values $\beta_l = 0.2$, $\beta_w = 0.3$, $\beta_h = \beta_{\max} = 0.4$ so that $\tilde{\beta} = 0.3$ and that each agent $i$ receives three signals: $l$, $w$ or $h$. In that case,

$$
\Gamma = \begin{pmatrix}
0.25 & 0.19 & 0.21 \\
0.19 & 0.33 & 0.23 \\
0.21 & 0.23 & 0.41 \\
\end{pmatrix}
$$

and the three eigenvalues are: $\lambda_1(\Gamma) = 0.76$, $\lambda_2(\Gamma) = 0.138$ and $\lambda_3(\Gamma) = 0.091$. Then:

$$
A = \begin{pmatrix}
0.485 & 0.166 & 0.859 \\
0.569 & 0.686 & -0.454 \\
0.664 & -0.709 & -0.238 \\
\end{pmatrix}, \quad A^{-1} = \begin{pmatrix}
0.485 & 0.569 & 0.664 \\
0.166 & 0.686 & -0.709 \\
0.859 & -0.454 & -0.238 \\
\end{pmatrix}
$$

We can now use Theorem 4 and state that, if $\lambda_{\max}(G) < \frac{1}{0.4\times0.76} \approx 3.289$, then if each individual $i$ receives the signal $s_i = l$, she provides a unique effort given by:

$$
x^* \{s_i = l\} = \alpha \left[ a_{il}^{-1}b(0.304, G) + a_{lw}a_{wl}^{-1}b(0.055, G) + a_{ih}a_{hl}^{-1}b(0.037, G) \\
+ a_{il}^{-1}b(0.304, G) + a_{lw}a_{wl}^{-1}b(0.055, G) + a_{ih}a_{hl}^{-1}b(0.037, G) \\
+ a_{il}^{-1}b(0.304, G) + a_{lw}a_{wl}^{-1}b(0.055, G) + a_{ih}a_{hl}^{-1}b(0.037, G) \\
\right]
$$

$$
= \alpha \left[ 0.235 b(0.304, G) + 0.028 b(0.055, G) + 0.737 b(0.037, G) \\
+ 0.276 b(0.304, G) + 0.114 b(0.055, G) - 0.39 b(0.037, G) \\
+ 0.322 b(0.304, G) - 0.118 b(0.055, G) - 0.204 b(0.037, G) \\
\right]
$$

Similar calculations can be done when each agent $i$ receives the signals $s_i = w$ and $s_i = h$.

### A.3 The Kronecker product

**Definition 4** Let $A$ be an $m_1 \times n_2$ matrix and $B$ an $n_1 \times n_2$ matrix. The Kronecker product (or tensor product) of $A$ and $B$ is defined as the $m_1 n_1 \times m_2 n_2$ matrix

$$
A \otimes B = \begin{pmatrix}
a_{11}B & \cdots & a_{1m_2}B \\
\vdots & \ddots & \vdots \\
a_{m_1}B & \cdots & a_{m_1m_2}B \\
\end{pmatrix}
$$

Here are some useful results:

**Proposition 7** Let $A$, $B$, $C$, and $D$ be matrices.
(i) If $A$ and $C$ are conformable and $B$ and $D$ are conformable, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD,$$

$$(A \otimes B)^T = A^T \otimes B^T.$$ 

(ii) If $A$ and $B$ are nonsingular, then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

(iii) Let the $n \times n$ matrix $A$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ and let the $m \times m$ matrix $B$ have eigenvalues $\mu_1, \ldots, \mu_m$. Then the $mn$ eigenvalues of $A \otimes B$ are

$$\lambda_1 \mu_1, \ldots, \lambda_1 \mu_m, \lambda_2 \mu_1, \ldots, \lambda_2 \mu_m, \ldots, \lambda_n \mu_1, \ldots, \lambda_n \mu_m.$$ 

A.4 Jordan decomposition

A.4.1 Some useful results on the Jordan canonical form

Diagonalizable matrices\(^{21}\) Consider a matrix $\Gamma \in \mathbb{C}^{n \times n}$, and denote its spectrum by $\sigma(\Gamma)$. Let $\lambda \in \sigma(\Gamma)$.

Definition 5

1. The algebraic multiplicity $\text{alg mult}_\Gamma(\lambda)$ of an eigenvalue $\lambda$ is the multiplicity of $\lambda$ as a root of the characteristic polynomial of $\Gamma$.

2. The geometric multiplicity $\text{geo mult}_\Gamma(\lambda)$ of an eigenvalue $\lambda$ is the dimension of the subspace spanned by the eigenvectors corresponding to $\lambda$.

The above definition suggests that it is possible for a repeated eigenvalue to be associated with more than one independent eigenvectors. This implies that $\text{geo mult}_\Gamma(\lambda) = \text{dim ker}(\Gamma - \lambda I)$. Different eigenvalues though correspond to linearly independent (“different”) eigenvectors. In fact, it can be shown (see, for example, Sadun, 2007, Theorem 4.8) that

$$1 \leq \text{geo mult}_\Gamma(\lambda) \leq \text{alg mult}_\Gamma(\lambda)$$ \hspace{1cm} (41)

This gives rise to the following definition

\(^{21}\)For a more thorough technical discussion as well as some intuition behind matrix diagonalizability, see Meyer (2001), Chap. 7.8, and Sadun (2007), Chap. 4.5.
Definition 6

1. An eigenvalue \( \lambda \) is called **simple** if \( \text{alg mult}_\Gamma(\lambda) = 1 \), that is \( \lambda \) is a non-repeated eigenvalue of \( \Gamma \).

2. An eigenvalue is called **semi-simple** if \( \text{alg mult}_\Gamma(\lambda) = \text{geo mult}_\Gamma(\lambda) \). A matrix is called **semi-simple** if all its eigenvalues are semi-simple.

3. An eigenvalue \( \lambda \) that is not semi-simple is called **defective** or **deficient** of (order of) deficiency \( d_\Gamma(\lambda) := \text{alg mult}_\Gamma(\lambda) = \text{geo mult}_\Gamma(\lambda) \). A matrix with one or more defective eigenvalues is called defective or deficient.

A well-known result in linear algebra is the following (see e.g. Meyer, 2001):

**Proposition 8** A matrix \( \Gamma \) is diagonalizable if and only if all its eigenvalues are semi-simple.

It follows directly from (41) that all simple eigenvalues are also semi-simple, but the converse, of course, will not hold in general. Hence

**Corollary 1** If all the eigenvalues of the matrix \( \Gamma \) are simple, then \( \Gamma \) is diagonalizable.

As the above discussion implies, the converse is not true: there are diagonalizable matrices with repeated eigenvalues. The identity matrix \( I_n \), for example, has only one eigenvalue, \( \lambda = 1 \), yet it is diagonalizable. In fact, any non-singular matrix can serve as its eigenbasis since \( \text{alg mult}_{I_n}(1) = \text{geo mult}_{I_n}(1) = n \).

Although deficient matrices are not diagonalizable, they can still be written as a similarity transformation of a matrix with their eigenvalues in the diagonal. This matrix will no longer be diagonal, but it will be block diagonal and have a special form, as discussed in the following subsection.

**Definition and motivation** The **Jordan canonical form** or simply **Jordan form** of a matrix \( \Gamma \in \mathbb{C}^{T \times T} \) is a \( T \times T \) block diagonal matrix similar to \( \Gamma \), that is, a matrix \( J_\Gamma \) satisfying \( \Gamma = A J_\Gamma A^{-1} \), that has the form

\[
J_\Gamma = \begin{bmatrix}
J_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \hat{J}_n
\end{bmatrix}
\]
that satisfies
\[ \Gamma = A \mathcal{J}_\Gamma A^{-1} \]
for some nonsingular matrix \(A\). The square sub-matrices \(\mathcal{J}_i, i = \{1, \ldots, S\}\) are called Jordan segments. There exists a Jordan segment \(\mathcal{J}_i\) of size \(\kappa_i \times \kappa_i\) for each distinct eigenvalue \(\lambda_i\) of \(\Gamma\). Hence, the number \(S\) of Jordan segments in the Jordan form of a matrix is equal to the number of distinct eigenvalues of that matrix, \(S = |\sigma(\Gamma)|\). The size of Jordan segment \(\mathcal{J}_i\) is equal to the algebraic multiplicity of the corresponding eigenvalue \(\lambda_i\), that is, \(\kappa_i = \text{alg mult}_\Gamma(\lambda_i)\), and its main diagonal consists of \(\lambda_i\)'s.

Each Jordan segment \(\mathcal{J}_i\) is a block diagonal matrix itself, consisting of Jordan blocks \(J_h\) that are square matrices of the form

\[
J_h(\lambda_i) = \begin{bmatrix}
\lambda_i & 1 & 0 \\
& \lambda_i & 1 \\
& & \ddots & 1 \\
& & & \lambda_i & 1 \\
& & & & \lambda_i
\end{bmatrix}
\]

The number \(Q\) of Jordan blocks in a Jordan segment \(\mathcal{J}_i\) is equal to the geometric multiplicity of the corresponding eigenvalue \(\lambda_i\),\(^{22}\) while the size of the largest Jordan block in segment \(\mathcal{J}_i\) is equal to \(\text{index}(\lambda_i)\). The structure of the Jordan form of a matrix, i.e. the number and the size of its Jordan segments and blocks, is uniquely determined by the elements of that matrix. This implies that the Jordan form of a matrix will be unique up to the ordering of the Jordan segments and the blocks within each segment.

The vectors comprising the basis \((A, A^{-1})\) of the similarity transformation are called the generalized eigenvectors of \(\Gamma\), and are, in general, not unique. The set of generalized eigenvectors corresponding to a Jordan block \(J_h(\lambda_i)\) constitute a Jordan chain. Notice that the eigenvector associated with \(\lambda_i\) will be a member of that Jordan chain, that is, eigenvectors are also generalized eigenvectors of a matrix. The converse is, however, not true.

**Functions of non-diagonal Jordan forms** Directly computing functions of nontrivial Jordan blocks can be cumbersome. Yet there exists an elegant expression for differentiable functions of Jordan blocks.

\(^{22}\)More specifically, it can be shown that the number of \(d\)-dimensional Jordan blocks in \(\mathcal{J}_i\) is given by

\[
\xi_d(\lambda_i) = r_d(\lambda_i) - 2r_{d-1}(\lambda_i) + r_{d+1}(\lambda_i)
\]

where \(d_s(\lambda_i) = \text{rank}((\Gamma - \lambda_i I)^s)\).
Proposition 9 For a $d_q \times d_q$ Jordan block $J_q(\lambda_i)$ associated with eigenvalue $\lambda_i$, and for a function $f(x)$ such that $f(\lambda_i)$, $f'(\lambda_i)$, ... , $f^{(k-1)}(\lambda_i)$ exist, $f(J_q(\lambda_i))$ is defined as:

$$f(J_h) = f \begin{pmatrix} \lambda_i & 1 & 0 \\ \lambda_i & 1 \\ \vdots & \ddots & 1 \\ 0 & \lambda_i & 1 \\ \lambda_i & \end{pmatrix} = f \begin{pmatrix} f(\lambda_i) & f'(\lambda_i) & f''(\lambda_i) & \cdots & f^{(d_q-1)}(\lambda_i) \\ f(\lambda_i) & f'(\lambda_i) & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & f'(\lambda_i) & f(\lambda_i) \\ f(\lambda_i) & \end{pmatrix}$$

A.4.2 Proof of Theorem 2

A more general approach Recall that in the $n-$player game, the equilibrium efforts of the agents as a function of the signal they receive are given by

$$\begin{pmatrix} x^*(\{s = 1\}) \\ \vdots \\ x^*(\{s = T\}) \end{pmatrix} = [I_{Tn} - \beta(\Gamma \otimes G)]^{-1} \begin{pmatrix} \hat{\alpha}_1 1_n \\ \vdots \\ \hat{\alpha}_T 1_n \end{pmatrix}$$

Let us rewrite the proof of Theorem 1 using the Jordan decomposition of matrix $\Gamma$ instead of its diagonal eigenvalue decomposition. We have:

$$\Gamma = A J_T A^{-1}$$

where $A$ is a non-singular $T \times T$ matrix and $J_T$ is the Jordan form of matrix $\Gamma$. Recall that since $G$ is a real symmetric matrix, it is diagonalizable with

$$G = C D_G C^{-1}$$

Let $\lambda_{\text{max}}(G)$ denote the spectral radius of matrix $G$, that is, $\lambda_{\text{max}}(G) := \max_{\lambda \in \sigma(G)} |\lambda|$. Then, assuming that $\beta \lambda_{\text{max}}(\Gamma \otimes G) = \beta \lambda_{\text{max}}(\Gamma) \lambda_{\text{max}} < 1$, it follows that

$$[I_{Tn} - \beta(\Gamma \otimes G)]^{-1} = [I_{Tn} - \beta(A \otimes C)(J_T \otimes D_G)(A^{-1} \otimes C^{-1})]^{-1}$$

$$= \sum_{k=0}^{+\infty} (A \otimes C)^\beta^k (J_T \otimes D_G)^k (A^{-1} \otimes C^{-1})$$

$$= (A \otimes C)^{+\infty} \sum_{k=0}^{+\infty} (\beta^k J_T^k \otimes D_G^k)(A^{-1} \otimes C^{-1})$$

(45)
The above expression differs from the respective expression in the paper in the term $\mathcal{J}_\Gamma^k$. Recall that $\mathcal{J}_\Gamma$ is a block diagonal matrix, consisting of Jordan blocks and zero matrices. Thus

$$
\mathcal{J}_\Gamma^k = \begin{bmatrix}
J_1^k & 0 & \cdots & 0 \\
0 & J_2^k & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & J_T^k
\end{bmatrix}
$$

An additional complication stems from calculating the terms $J_q^k$. For Jordan blocks of diagonalizable matrices, or Jordan blocks of deficient matrices associated with semi-simple eigenvalues, it is easy to calculate these terms since $J_q^k$ will be a degenerate $1 \times 1$ matrix given by:

$$
J_q^k = \begin{bmatrix}
\lambda_q^k \\
\lambda_q^k \\
\vdots \\
\lambda_q^k \\
0 \\
\vdots \\
\lambda_q^k \\
\lambda_q^k \\
\lambda_q^k
\end{bmatrix}
$$

If, however, $\Gamma$ is not diagonalizable, its Jordan form will contain at least one non-diagonal Jordan block of the form given in (42). In that case, we need to resort to formula (43), letting $f(x) = x^k$,

$$
J_q^k = \begin{bmatrix}
\lambda_q^k & k\lambda_q^k \lambda_q^k \ldots \frac{(k-1)k\lambda_q^k}{2} \ldots \frac{(k-(d_q-2))k\lambda_q^k}{(d_q-2)!} \ldots \\
\lambda_q^k & k\lambda_q^k \lambda_q^k \ldots \frac{(k-1)k\lambda_q^k}{2} \lambda_q^k \lambda_q^k \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \lambda_q^k \\
0 & \lambda_q^k \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \lambda_q^k \\
\lambda_q^k & (k)\lambda_q^k \lambda_q^k \ldots \lambda_q^k \lambda_q^k \lambda_q^k
\end{bmatrix}
$$

or more compactly

$$
J_q^k = \begin{bmatrix}
\lambda_q^k & \binom{k}{1} \lambda_q^k \lambda_q^k \ldots \binom{k}{d_q-1} \lambda_q^k \lambda_q^k \lambda_q^k \\
\lambda_q^k & \binom{k}{2} \lambda_q^k \lambda_q^k \lambda_q^k \ldots \lambda_q^k \lambda_q^k \lambda_q^k \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \lambda_q^k \\
0 & \lambda_q^k \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \lambda_q^k \\
\lambda_q^k & (k)\lambda_q^k \lambda_q^k \ldots \lambda_q^k \lambda_q^k
\end{bmatrix}
$$

It follows then that $J_q^k$, and thus $\mathcal{J}_\Gamma^k$, will not be diagonal matrices as $D_\Gamma^k$ is. As a result, the breakdown of the vector of equilibrium efforts $x^*$ into Katz-Bonacich measures is complicated and to obtain the expression (29), we will first consider the case of a $3 \times 3$ information matrix $\Gamma$ (3 states of the world and 3 signals).
The case of a 3 × 3 information matrix \( \Gamma \)  

In order to see how expression (29) changes when the information matrix \( \Gamma \) is non-diagonalizable, we first start with an example. Suppose that there are three possible values for the signal, leading to a 3 × 3 information matrix \( \Gamma \).

Assume that \( \Gamma \) possesses a simple eigenvalue, \( \lambda_1 \), and a defective double eigenvalue, \( \lambda_2 = \lambda_3 \). Hence \( \Gamma \) will not be diagonalizable. Yet, as discussed above, a diagonal Jordan decomposition will still exist and given by:

\[
J_\Gamma = \begin{pmatrix}
J_{1(1\times 1)} & 0_{(1\times 2)} \\
0_{(2\times 1)} & J_{2(2\times 2)}
\end{pmatrix} = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 1 \\
0 & 0 & \lambda_3
\end{pmatrix}
\]

For notational convenience, it will useful to relabel the eigenvalues of \( \Gamma \) so that \( \hat{\lambda}_q \) denotes the eigenvalue associated with the \( q \)-th Jordan block, that is \( \hat{\lambda}_1 := \lambda_1 = \lambda_2 \) and \( \hat{\lambda}_2 := \lambda_3 \). Thus,

\[
J_\Gamma = \begin{pmatrix}
\hat{\lambda}_1 & 0 & 0 \\
0 & \hat{\lambda}_2 & 1 \\
0 & 0 & \hat{\lambda}_3
\end{pmatrix}
\]

Using (46), we have:

\[
J_{\Gamma}^k = \begin{pmatrix}
\hat{\lambda}_1^k & 0 & 0 \\
0 & \hat{\lambda}_2^k & k\hat{\lambda}_2^{k-1} \\
0 & 0 & \hat{\lambda}_3^k
\end{pmatrix}
\]

Then, by using standard matrix algebra, it is easy to show that

\[
\sum_{k=0}^{+\infty} (\beta^k J_\Gamma^k \otimes D_G^k) = \begin{pmatrix}
\sum_k \beta^k \hat{\lambda}_1^k D_G^k & 0 & 0 \\
0 & \sum_k \beta^k \hat{\lambda}_2^k D_G^k & \sum_k k \beta^k \hat{\lambda}_2^{k-1} D_G^k \\
0 & 0 & \sum_k \beta^k \hat{\lambda}_3^k D_G^k
\end{pmatrix}
\]  \hspace{1cm} (47)

Recall that if \( \Gamma \) is diagonalizable, then matrix \( \sum_{k=0}^{+\infty} (\beta^k D_G^k \otimes D_G^k) \) will be diagonal, as shown in the proof of Theorem 1. Here, in our example, this will not be the case since the term \( \sum_k k \beta^k \hat{\lambda}_2^{k-1} D_G^k \) appears in the \((2, 3)\) block of this matrix. This term is a source of potential concern since, apart from complicating the algebra, it is not straightforward to interpret this expression as some measure of Katz-Bonacich centrality. It will, however, turn out to be the case. Indeed, taking into account (47), expression (45) can be written as:

\[
[I_{Tn} - \beta (\Gamma \otimes G)]^{-1} =
\]

54
\[
\begin{pmatrix}
  a_{11}C & a_{12}C & a_{13}C \\
  a_{21}C & a_{22}C & a_{23}C \\
  a_{31}C & a_{32}C & a_{33}C
\end{pmatrix}
\begin{pmatrix}
  \sum_k \beta^k \lambda_1^k D_G^k & 0 & 0 \\
  0 & \sum_k \beta^k \lambda_2^k D_G^k & \sum_k \beta^k \lambda_1^{k-1} D_G^k \\
  0 & 0 & \sum_k \beta^k \lambda_2^{k-1} D_G^k
\end{pmatrix}
\times
\begin{pmatrix}
  a_{11}^{(-1)} C^{-1} & a_{12}^{(-1)} C^{-1} & a_{13}^{(-1)} C^{-1} \\
  a_{21}^{(-1)} C^{-1} & a_{22}^{(-1)} C^{-1} & a_{23}^{(-1)} C^{-1} \\
  a_{31}^{(-1)} C^{-1} & a_{32}^{(-1)} C^{-1} & a_{33}^{(-1)} C^{-1}
\end{pmatrix}
\end{align*}

\[
= \begin{pmatrix}
  a_{11} \sum_k \beta^k \lambda_1^k D_G^k & a_{12} \sum_k \beta^k \lambda_2^k D_G^k & a_{13} \sum_k \beta^k \lambda_1^{k-1} D_G^k \\
  a_{21} \sum_k \beta^k \lambda_1^k D_G^k & a_{22} \sum_k \beta^k \lambda_2^k D_G^k & a_{23} \sum_k \beta^k \lambda_2^{k-1} D_G^k \\
  a_{31} \sum_k \beta^k \lambda_1^k D_G^k & a_{32} \sum_k \beta^k \lambda_2^k D_G^k & a_{33} \sum_k \beta^k \lambda_2^{k-1} D_G^k
\end{pmatrix}
\times
\begin{pmatrix}
  a_{11}^{(-1)} C^{-1} & a_{12}^{(-1)} C^{-1} & a_{13}^{(-1)} C^{-1} \\
  a_{21}^{(-1)} C^{-1} & a_{22}^{(-1)} C^{-1} & a_{23}^{(-1)} C^{-1} \\
  a_{31}^{(-1)} C^{-1} & a_{32}^{(-1)} C^{-1} & a_{33}^{(-1)} C^{-1}
\end{pmatrix}
\]

\[
\begin{align*}
&= \begin{pmatrix}
  a_{11} a_{11}^{(-1)} M_0 \left( \tilde{\lambda}_1, G \right) + a_{12} a_{21}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right) + a_{13} a_{31}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right) + a_{13} a_{31}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right) \\
  a_{21} a_{11}^{(-1)} M_0 \left( \tilde{\lambda}_1, G \right) + a_{22} a_{21}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right) + a_{23} a_{31}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right) + a_{23} a_{31}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right) \\
  a_{31} a_{11}^{(-1)} M_0 \left( \tilde{\lambda}_1, G \right) + a_{32} a_{21}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right) + a_{33} a_{31}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right) + a_{33} a_{31}^{(-1)} M_0 \left( \tilde{\lambda}_2, G \right)
\end{pmatrix}
\end{align*}
\]

\[
\begin{pmatrix}
  \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{h} \sum_{v=D_{q-1}+v_1}^{h} a_{1v} a_{h1}^{(-1)} M_{h-v} \left( \tilde{\lambda}_q, G \right) & \ldots & \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{h} \sum_{v=D_{q-1}+v_1}^{h} a_{1v} a_{h3}^{(-1)} M_{h-v} \left( \tilde{\lambda}_q, G \right) \\
  \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{h} \sum_{v=D_{q-1}+v_1}^{h} a_{2v} a_{h1}^{(-1)} M_{h-v} \left( \tilde{\lambda}_q, G \right) & \ldots & \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{h} \sum_{v=D_{q-1}+v_1}^{h} a_{2v} a_{h3}^{(-1)} M_{h-v} \left( \tilde{\lambda}_q, G \right) \\
  \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{h} \sum_{v=D_{q-1}+v_1}^{h} a_{3v} a_{h1}^{(-1)} M_{h-v} \left( \tilde{\lambda}_q, G \right) & \ldots & \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{h} \sum_{v=D_{q-1}+v_1}^{h} a_{3v} a_{h3}^{(-1)} M_{h-v} \left( \tilde{\lambda}_q, G \right)
\end{pmatrix}
\]

(48)
where

\[
M_0(\lambda_q \beta, G) := \sum_{k=0}^{+\infty} \beta^k \lambda_q G^k = \Lambda(\lambda_q \beta, G)
\]

\[
M_1(\lambda_q \beta, G) := \sum_{k=0}^{+\infty} k \beta^k \lambda_q^{k-1} G^k
\]

\[D_q := \sum_{i=1}^q d_i\]

where \(d_i\) denotes the size of the \(i\)th Jordan block. It can be observed that

\[
M_0(\lambda_q \beta, G)1_n = \Lambda(\lambda_q \beta, G)1_n = b(\lambda_q \beta, G)
\]

The interpretation of \(M_1(\lambda_q \beta, G)\) is not as straightforward. Yet it can be shown that \(M_1(\lambda_q \beta, G)\) leads to weighted Katz-Bonacich centrality. Indeed, we have the following result.

**Lemma 5** Let \(u_{n,1}(\lambda_q) := (I_n - \lambda_q \beta G)^{-1} \beta G 1_n\), and denote the \(u_{n,1}(\lambda_q)\)-weighted Katz-Bonacich centrality by \(b_{u_1}\). Then,

\[
M_1(\lambda_q \beta, G)1_n = b_{u_1}(\lambda_q \beta, G)
\]

**Proof:** Recall that, by definition,

\[
b(\lambda_q \beta, G) := \sum_{k=0}^{+\infty} (\lambda_q \beta)^k G^k 1_n
\]

If \(\beta \lambda_q < 1/\lambda_{\text{max}}(G)\), then

\[
b(\lambda_q \beta, G) = (I_n - \lambda_q \beta G)^{-1} 1_n
\]

Hence, using the definition of Katz-Bonacich centrality in (51), we have:

\[
\frac{\partial b(\lambda_q \beta, G)}{\partial \lambda_q} = \frac{\partial}{\partial \lambda_q} \left[ \sum_{k=0}^{+\infty} (\lambda_q \beta)^k G^k 1_n \right] = \sum_{k=0}^{+\infty} k \beta^k \lambda_q^{k-1} G^k 1_n = M_1(\lambda_q \beta, G)1_n
\]
Similarly, by the alternative expression for Katz-Bonacich given in (52), we have:

\[
\frac{\partial b(\hat{\lambda}_q \beta, G)}{\partial \lambda_q} = \frac{\partial}{\partial \lambda_q} \left[(I_n - \hat{\lambda}_q \beta G)^{-1}1_n\right]
\]

\[
= -(I_n - \hat{\lambda}_q \beta G)^{-1} \frac{\partial}{\partial \lambda_q} (I_n - \hat{\lambda}_q \beta G)^{-1}1_n
\]

\[
= -(I_n - \hat{\lambda}_q \beta G)^{-1}(-\beta G)(I_n - \hat{\lambda}_q \beta G)^{-1}1
\]

\[
= (I_n - \hat{\lambda}_q \beta G)^{-1} \beta G (I_n - \hat{\lambda}_q \beta G)^{-1}1
\]

\[
= (I_n - \hat{\lambda}_q \beta G)^{-1} (I_n - \hat{\lambda}_q \beta G)^{-1} \beta G 1_n
\]

\[
= (I_n - \hat{\lambda}_q \beta G)^{-1} u_{n,1}
\]

\[
= b_{u_1} (\hat{\lambda}_q \beta, G)
\]

(54)

where \(u_{n,1}(\hat{\lambda}_q) := (I_n - \hat{\lambda}_q \beta G)^{-1} \beta G 1_n\), and the fifth equality follows from the fact that \((I_n - \hat{\lambda}_q \beta G)^{-1}\) and \(G\) commute. Let us show that, indeed, matrices \(G\) and \((I_n - \hat{\lambda}_q \beta G)^{-1}\) are commutative by the following lemma.

**Lemma 6** Let \(A\) be a nonsingular matrix and let \(B\) be a conformable matrix that commutes with \(A\). The \(B\) also commutes with \(A^{-h}\), for \(h \in \mathbb{N}\).

**Proof:** Let us start by showing that \(A^{-1}\) and \(B\) commute. By using the assumption that \(A\) and \(B\) commute, and pre- and post-multiplying by \(A^{-1}\), we obtain:

\[
A B = B A
\]

\[
A^{-1} (A B) A^{-1} = A^{-1} (B A) A^{-1}
\]

\[
B A^{-1} = A^{-1} B
\]

(55)

It is now straightforward to show that matrices \(A^{-h}\) and \(B\) are commutative. Indeed

\[
A^{-h} B = A^{-h+1} A^{-1} B
\]

\[
= A^{-h+1} B A^{-1}
\]

\[
= A^{-h+2} A^{-1} B A^{-1}
\]

\[
= A^{-h+2} B A^{-2}
\]

\[
= ...
\]

\[
= A^{-1} B A^{-h+1}
\]

\[
= B A^{-h}
\]
where we have used (55). □

Let us go back to the proof of Lemma 5. We have shown that

\[ \frac{\partial \mathbf{b}(\hat{\lambda}_q, \mathbf{G})}{\partial \lambda_q} = \mathbf{b}_{u_1}(\hat{\lambda}_q, \mathbf{G}) \]

Therefore, equalities (53) and (54) imply that:

\[ \mathbf{M}_1(\hat{\lambda}_q, \mathbf{G}) \mathbf{1}_n = \mathbf{b}_{u_1}(\hat{\lambda}_q, \mathbf{G}) \]

which is the statement of the lemma. □

We have thus showed that \( \mathbf{M}_1(\hat{\lambda}_q, \mathbf{G}) \mathbf{1}_n = \mathbf{b}_{u_1}(\hat{\lambda}_q, \mathbf{G}) \). Substituting (48) into (44), and taking into account (49), the vector of (stacked) equilibrium efforts can be written as:

\[
\begin{pmatrix}
  \mathbf{x}^*(\{s = 1\}) \\
  \mathbf{x}^*(\{s = 2\}) \\
  \mathbf{x}^*(\{s = 3\})
\end{pmatrix} = \begin{pmatrix}
  \sum_{t=1}^{3} \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{D_{q-1}+d_q} \sum_{\nu=D_{q-1}+1}^{h} a_1 a_{d_{ht}} (-1)^{\nu-1} \mathbf{M}_{h-\nu}(\hat{\lambda}_q, \mathbf{G}) \\
  \sum_{t=1}^{3} \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{D_{q-1}+d_q} \sum_{\nu=D_{q-1}+1}^{h} a_2 a_{d_{ht}} (-1)^{\nu-1} \mathbf{M}_{h-\nu}(\hat{\lambda}_q, \mathbf{G}) \\
  \sum_{t=1}^{3} \sum_{q=1}^{2} \sum_{h=D_{q-1}+d_q}^{D_{q-1}+d_q} \sum_{\nu=D_{q-1}+1}^{h} a_3 a_{d_{ht}} (-1)^{\nu-1} \mathbf{M}_{h-\nu}(\hat{\lambda}_q, \mathbf{G})
\end{pmatrix} \]

(56)

It can thus be seen that the equilibrium strategies are a linear combination of unweighted and weighted Katz-Bonacich centralities.

**Generalization to an arbitrary information matrix** It can be shown that above conclusion carries over to the more general case of a \( T \times T \) matrix \( \Gamma \) with any number of defective eigenvalues of arbitrary deficiency (Theorem 2). Indeed, assume that the Jordan form of \( \Gamma \) consists of \( Q \) Jordan blocks, \( J_q(\hat{\lambda}_q) \), \( q \in \{1, \ldots, Q\} \). Using the same technique as above, it can be seen that the \((\tau, t)\)-block of matrix \( [\mathbf{I}_{Tn} - \beta(\Gamma \otimes \mathbf{G})]^{-1} \) can be written as:

\[
[I_{Tn} - \beta(\Gamma \otimes \mathbf{G})]^{-1}_{(\tau, t)} = \sum_{q=1}^{Q} \sum_{D_{q-1}+d_q}^{D_{q-1}+d_q} \sum_{D_{q-1}+1}^{h} \sum_{D_{q-1}+1}^{D_{q-1}+1} a_{\tau \nu} a_{d_{ht}} (-1)^{\nu-1} \mathbf{M}_{h-\nu}(\hat{\lambda}_q, \mathbf{G})
\]

(57)

where

\[
\mathbf{M}_{h-\nu}(\hat{\lambda}_q, \mathbf{G}) := \sum_{k=0}^{+\infty} \binom{k}{h-\nu} \hat{\lambda}_q^{h-\nu} \beta^k \mathbf{G}^k
\]

(58)

and let \( D_Q, d_q \) and \( \hat{\lambda}_q \) defined as above. The following result is useful in obtaining the desired characterization of the equilibrium efforts.\(^{23}\)

\(^{23}\)Observe that in order to keep notation as simple as possible and since there is no risk of confusion, similarly to Lemma 5, we define

\[
\mathbf{b}_{u_{h-\nu}}(\hat{\lambda}_q, \mathbf{G}) := \mathbf{b}_{u_{h-\nu}}(\hat{\lambda}_q, \mathbf{G})
\]

58
Lemma 7 For \( h \in N \), the matrix \( M_h(\hat{\lambda}q, G) \) can be mapped into a vector of \( u_{n,h}(\hat{\lambda}q) \)-weighted Katz-Bonacich centralities, with \( u_{n,h}(\hat{\lambda}q) := (I_n - \beta G)^{-h} \beta^h G^h 1_n \), as follows:

\[
M_h(\hat{\lambda}q, G) 1_n = b_{u_{h}}(\hat{\lambda}q, G) \tag{59}
\]

Hence, the \( h \)-th order derivative of the unweighted Katz-Bonacich centrality measure \( b(\hat{\lambda}q, G) \) with respect to \( \hat{\lambda}q \) is still a weighted Katz-Bonacich centrality. More generally, for \( m, \nu \in N \), the \( m \)-th order derivative of the weighted Katz-Bonacich centrality \( b_{u_{\nu}}(\hat{\lambda}q, G) \) with respect to \( \hat{\lambda}q \) is still a weighted Katz-Bonacich centrality, albeit with a different weight.

Proof: Using (51), we can calculate the second derivative of the Katz-Bonacich centrality measure as follows:

\[
\frac{\partial^2 b(\hat{\lambda}q, G)}{\partial \hat{\lambda}q^2} = \frac{\partial b_{u_{1}}(\hat{\lambda}q, G)}{\partial \hat{\lambda}q} = \frac{\partial}{\partial \hat{\lambda}q} \left[ \sum_{k=0}^{+\infty} k \hat{\lambda}^k_q G^k 1_n \right] = \sum_{k=0}^{+\infty} (k-1) \hat{\lambda}^{k-2}_q G^k 1_n = 2 M_2(\hat{\lambda}q, G) 1_n
\]

Similarly, using (52), we get

\[
\frac{\partial^2 b(\hat{\lambda}q, G)}{\partial \hat{\lambda}q^2} = \frac{\partial b_{u_{2}}(\hat{\lambda}q, G)}{\partial \hat{\lambda}q} = \frac{\partial}{\partial \hat{\lambda}q} \left[ \sum_{k=0}^{+\infty} k \hat{\lambda}^k_q G^k 1_n \right]
\]

\[
= 2 \left[ (I_n - \hat{\lambda}q G)^{-1}(I_n - \hat{\lambda}q G)^{-1} \beta G 1_n \right] = 2 (I_n - \hat{\lambda}q G)^{-1}(I_n - \hat{\lambda}q G)^{-2} \beta^2 G^2 1_n
\]

59
where $u_2 := (I_n - \hat{\lambda}_q \beta G)^{-2} \beta^2 G^2 1_n$, and the fifth equality follows from the commutation of $(I_n - \hat{\lambda}_q \beta G)^{-h}$ and $G$ for $h \in \mathbb{N}$ (see Lemma 6).

The general pattern for the $h$-th order derivative of each expression of the Katz-Bonacich centrality $b(\hat{\lambda}_q \beta, G)$ starts now to emerge. Using (51), we obtain:

$$
\frac{\partial^h b(\hat{\lambda}_q \beta, G)}{\partial \hat{\lambda}_q} = \sum_{k=0}^{+\infty} \frac{(h!)^k}{h!} \lambda_q^{k-h} \beta^k G^k 1_n
$$

where the last equality follows from the definition of $M_h$ given in (58). Similarly, starting from (52) and taking into account that $G$ and $(I_n - \hat{\lambda}_q \beta G)^{-h}$ commute, it can be shown that

$$
\frac{\partial^h b(\hat{\lambda}_q \beta, G)}{\partial \hat{\lambda}_h} = (I_n - \hat{\lambda}_q \beta G)^{-h-1}(h!) \beta^h G^h 1_n
$$

where $u_h := (I_n - \hat{\lambda}_q \beta G)^{-h} \beta^h G^h 1_n$. Now notice that (60) and (61) imply that

$$
M_h(\hat{\lambda}_q \beta, G) 1_n = b u_h(\hat{\lambda}_q \beta, G)
$$

which is precisely equation (59).

We have shown that $M_h(\hat{\lambda}_q \beta, G) 1_n = b u_h(\lambda_t \beta, G)$, that is, $M_h(\hat{\lambda}_q \beta, G)$ can be mapped into a vector of $u_h$-weighted Katz-Bonacich centralities, with $u_h := (I_n - \hat{\lambda}_q \beta G)^{-h} \beta^h G^h 1_n$.

Then, substituting (57) into (44) and applying Lemma 7 yields

$$
\begin{pmatrix}
\vdots \\
\sum_{s=1}^Q \sum_{h=D_{q-1}+d_q}^{D_q-1+d_q} \sum_{\nu=D_{q-1}+1}^{D_q-1+1} a_{\nu \nu} \alpha^{(1)} \beta_{\nu \nu} m_{h-\nu}(\hat{\lambda}_q \beta, G) \vdots \\
\vdots \\
\vdots
\end{pmatrix} = \begin{pmatrix}
\vdots \\
\sum_{s=1}^Q \sum_{h=D_{q-1}+d_q}^{D_q-1+d_q} \sum_{\nu=D_{q-1}+1}^{D_q-1+1} a_{\nu \nu} \alpha^{(1)} M_{h-\nu}(\hat{\lambda}_q \beta, G) \vdots \\
\vdots \\
\vdots
\end{pmatrix} \begin{pmatrix}
\vdots \\
\hat{\alpha}_s 1_n \\
\vdots
\end{pmatrix}
$$
\[
\begin{align*}
&= \left( \sum_{l=1}^{T} \hat{\alpha}_l \left[ \sum_{q=1}^{Q} \sum_{h=q-1+1}^{D_{q-1}+d_q} \sum_{\nu=q-1+1}^{h} a_{\tau \nu} a_{h \tau}^{(-1)} M_{h \nu}(\hat{\lambda}_q \beta, G) \right] 1_n \right) \\
&= \left( \sum_{l=1}^{T} \hat{\alpha}_l \sum_{q=1}^{Q} \sum_{h=q-1+1}^{D_{q-1}+d_q} \sum_{\nu=q-1+1}^{h} a_{\tau \nu} a_{h \tau}^{(-1)} b_{h \nu}(\hat{\lambda}_q \beta, G) \right)
\end{align*}
\]

and thus expression (34) in Theorem 2 is obtained.

Hence a generalized version of Theorem 1 (given by Theorem 2), applicable to any matrix \( \Gamma \), provides an expression for the equilibrium efforts that is a linear combination of weighted Katz-Bonacich centralities. In the light of the above discussion, deriving explicitly such expression seems straightforward, albeit quite tedious in terms of algebra and notation, since it must take into account the deficiency of the eigenvalues of \( \Gamma \).

### A.5 Proofs

#### Proof of Lemma 1:

We have:

\[
P(\{s_i = l\}) = P(\{s_i = l\} | \{\alpha = \alpha_l\}) P(\{\alpha = \alpha_l\}) + P(\{s_i = l\} | \{\alpha = \alpha_h\}) P(\{\alpha = \alpha_h\})
\]

\[
= q (1 - p) + (1 - q) p
\]

and

\[
P(\{s_i = h\}) = P(\{s_i = h\} | \{\alpha = \alpha_l\}) P(\{\alpha = \alpha_l\}) + P(\{s_i = h\} | \{\alpha = \alpha_h\}) P(\{\alpha = \alpha_h\})
\]

\[
= (1 - q) (1 - p) + q p
\]

We have:

\[
P(\{\alpha = \alpha_l\} \cap \{s_j = l\} | \{s_i = l\}) = \frac{P(\{s_j = l\} \cap \{s_i = l\} \cap \{\alpha = \alpha_l\})}{P(\{s_i = l\})}
\]

\[
= \frac{P(\{s_j = l\} | \{s_i = l\} \cap \{\alpha = \alpha_l\}) \ P(\{s_i = l\} | \{\alpha = \alpha_l\}) \ P(\{\alpha = \alpha_l\})}{P(\{s_i = l\})}
\]

\[
= \frac{q^2 (1 - p)}{q (1 - p) + (1 - q) p}
\]

61
and thus

\[
\mathbb{P}(\{\alpha = \alpha_l\} \cap \{s_j = h\} \mid \{s_i = l\}) = \frac{q(1-p)(1-q)}{q(1-p) + (1-q)p}
\]

\[
\mathbb{P}(\{\alpha = \alpha_h\} \cap \{s_j = l\} \mid \{s_i = l\}) = \frac{p(1-q)^2}{q(1-p) + (1-q)p}
\]

\[
\mathbb{P}(\{\alpha = \alpha_h\} \cap \{s_j = h\} \mid \{s_i = l\}) = \frac{(1-q)pq}{q(1-p) + (1-q)p}
\]

Similarly,

\[
\mathbb{P}(\{\alpha = \alpha_l\} \cap \{s_j = l\} \mid \{s_i = h\}) = \frac{(1-q)(1-p)q}{qp + (1-q)(1-p)}
\]

\[
\mathbb{P}(\{\alpha = \alpha_l\} \cap \{s_j = h\} \mid \{s_i = h\}) = \frac{(1-p)(1-q)^2}{qp + (1-q)(1-p)}
\]

\[
\mathbb{P}(\{\alpha = \alpha_h\} \cap \{s_j = l\} \mid \{s_i = h\}) = \frac{qp(1-q)}{qp + (1-q)(1-p)}
\]

\[
\mathbb{P}(\{\alpha = \alpha_h\} \cap \{s_j = h\} \mid \{s_i = h\}) = \frac{pq^2}{qp + (1-q)(1-p)}
\]

As a result,

\[
\gamma_l = \mathbb{P}(\{s_j = l\} \mid \{s_i = l\}) = \mathbb{P}(\{\alpha = \alpha_l\} \cap \{s_j = l\} \mid \{s_i = l\}) + \mathbb{P}(\{\alpha = \alpha_h\} \cap \{s_j = l\} \mid \{s_i = l\}) = \frac{(1-p)q^2 + p(1-q)^2}{q(1-p) + (1-q)p}
\]

\[
1 - \gamma_l = \mathbb{P}(\{s_j = h\} \mid \{s_i = l\}) = \frac{q(1-q)}{q(1-p) + (1-q)p}
\]

\[
1 - \gamma_h = \mathbb{P}(\{s_j = l\} \mid \{s_i = h\}) = \mathbb{P}(\{\alpha = \alpha_l\} \cap \{s_j = l\} \mid \{s_i = h\}) + \mathbb{P}(\{\alpha = \alpha_h\} \cap \{s_j = l\} \mid \{s_i = h\}) = \frac{q(1-q)}{qp + (1-q)(1-p)}
\]

\[
\gamma_h = \mathbb{P}(\{s_j = h\} \mid \{s_i = h\}) = \frac{(1-p)(1-q)^2 + pq^2}{qp + (1-q)(1-p)}
\]

This proves the results.
Proof of Proposition 2: The first-order conditions are given by (22), i.e.

\[ x = [I_{2n} - \beta \Gamma \otimes G]^{-1} \hat{\alpha} \]

where \( x = \left( \begin{array}{c} \hat{x} \\ \hat{y} \end{array} \right) \) and \( \hat{\alpha} = \left( \begin{array}{c} \hat{\alpha}_l \mathbf{1}_n \\ \hat{\alpha}_h \mathbf{1}_n \end{array} \right). \)

First, let us show that \( I_{2n} - \beta \Gamma \otimes G \) is non-singular. This is true if \( 0 < \beta < 1/\lambda_{\text{max}}(G) \) holds true. Indeed, since \( \Gamma \) is a stochastic matrix and thus \( \lambda_{\text{max}}(\Gamma) = 1 \), then \( \lambda_{\text{max}}(\Gamma \otimes G) = \lambda_{\text{max}}(\Gamma) \lambda_{\text{max}}(G) = \lambda_{\text{max}}(G) \). Therefore, if \( 0 < \beta < 1/\lambda_{\text{max}}(G) \) holds true, \( I_{2n} - \beta \Gamma \otimes G \) is non-singular. This shows that the system above has a unique solution and thus there exists a unique Nash-Bayesian equilibrium. This solution is interior since we have assumed that \( 0 < \alpha_l < \alpha_h \), which implies that \( \hat{\alpha}_l > 0 \) and \( \hat{\alpha}_h > 0 \).

Second, to show that the equilibrium action of each agent \( i \) is a linear function of the Katz-Bonacich centrality measures \( b_i(\beta, G) \) and \( b_i((\gamma_h + \gamma_l - 1) \beta, G) \), let us diagonalize the two main matrices \( \Gamma \) and \( G \).

Since \( \Gamma \) is a \( 2 \times 2 \) stochastic matrix, it can be diagonalized as follows:

\[ \Gamma = A \; D_{\Gamma} \; A^{-1} \]

where

\[ D_{\Gamma} = \begin{pmatrix} \lambda_1(\Gamma) & 0 \\ 0 & \lambda_2(\Gamma) \end{pmatrix} \]

and where \( \lambda_1(\Gamma) \geq \lambda_2(\Gamma) \) are the eigenvalues of \( \Gamma \). Observe that \( \Gamma \) is given by

\[ \Gamma = \begin{pmatrix} \gamma_l & 1 - \gamma_l \\ 1 - \gamma_h & \gamma_h \end{pmatrix} \]

It is easily verified that the two eigenvalues are \( \lambda_1(\Gamma) \equiv \lambda_{\text{max}}(\Gamma) = 1 \) and \( \lambda_2(\Gamma) = \gamma_h + \gamma_l - 1 \) and that

\[ A = \begin{pmatrix} 1 & - (1 - \gamma_l) / (1 - \gamma_h) \\ 1 & 1 \end{pmatrix} \]

\[ A^{-1} = \begin{pmatrix} 1 - \gamma_h \\ 2 - \gamma_l - \gamma_h \end{pmatrix} \begin{pmatrix} 1 & (1 - \gamma_l) / (1 - \gamma_h) \\ -1 & 1 \end{pmatrix} \]

Thus

\[ D_{\Gamma} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_h + \gamma_l - 1 \end{pmatrix} \]

Moreover, the \( n \times n \) network adjacency matrix \( G \) is symmetric and therefore can be diagonalized as follows:

\[ G = CD_G C^{-1} \]

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where
\[
D_G = \begin{pmatrix}
\lambda_1(G) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_n(G)
\end{pmatrix}
\]  
(63)

and where \( \lambda_{\text{max}}(G) := \lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G) \) are the eigenvalues of \( G \).

In this context, the equilibrium system (22) can be written as:

\[
x^* = \left[ I_{2n} - \beta \Gamma \otimes G \right]^{-1} \tilde{\alpha}
= \left[ I_{2n} - \beta \left( A D_\Gamma A^{-1} \right) \otimes \left( CD_G C^{-1} \right) \right]^{-1} \tilde{\alpha}
= \left[ I_{2n} - \beta (A \otimes C) (D_\Gamma \otimes D_G) (A^{-1} \otimes C^{-1}) \right]^{-1} \tilde{\alpha}
\]

We have:

\[
\left[ I_{2n} - \beta (A \otimes C) (D_\Gamma \otimes D_G) (A^{-1} \otimes C^{-1}) \right]^{-1}
= (A \otimes C) \sum_{k=1}^{+\infty} \beta^k (D_\Gamma \otimes D_G)^k (A^{-1} \otimes C^{-1})
\]

where we have used the properties of the Kronecker product (see Appendix A.3).

Using (62), we have:

\[
D_\Gamma \otimes D_G = \begin{pmatrix}
D_G & 0 \\
0 & (\gamma_h + \gamma_l - 1) D_G
\end{pmatrix}
\]

where \( D_G \) is defined by (63). Thus,

\[
(D_\Gamma \otimes D_G)^k = \begin{pmatrix}
D_G^k & 0 \\
0 & (\gamma_h + \gamma_l - 1)^k D_G^k
\end{pmatrix}
\]

Let us have the following notations:

\[
\Lambda_1 := \sum_{k=1}^{+\infty} \beta^k D_G^k \quad \text{and} \quad \Lambda_2 := \sum_{k=1}^{+\infty} \beta^k (\gamma_h + \gamma_l - 1)^k D_G^k
\]
Then,

\[
(A \otimes C) \sum_{k=1}^{+\infty} \beta^k (D_T \otimes D_G)^k (A^{-1} \otimes C^{-1})
\]

\[
= (A \otimes C) \begin{pmatrix}
\Lambda_1 & 0 \\
0 & \Lambda_2
\end{pmatrix} (A^{-1} \otimes C^{-1})
\]

\[
= \left( \frac{1 - \gamma_h}{2 - \gamma_l - \gamma_h} \right) \begin{pmatrix}
C & \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) C \\
C & C
\end{pmatrix} \begin{pmatrix}
\Lambda_1 & 0 \\
0 & \Lambda_2
\end{pmatrix} \begin{pmatrix}
C^{-1} & \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) C^{-1} \\
- C^{-1} & \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) C^{-1}
\end{pmatrix}
\]

\[
= \left( \frac{1 - \gamma_h}{2 - \gamma_l - \gamma_h} \right) \begin{pmatrix}
\Lambda_1 C^{-1} & \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) \Lambda_1 C^{-1} \\
- \Lambda_2 C^{-1} & \Lambda_2 C^{-1}
\end{pmatrix}
\]

\[
= \left( \frac{1 - \gamma_h}{2 - \gamma_l - \gamma_h} \right) \begin{pmatrix}
CA_1 C^{-1} + \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) CA_2 C^{-1} & \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) CA_1 C^{-1} - \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) CA_2 C^{-1} \\
CA_1 C^{-1} - CA_2 C^{-1} & \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) CA_1 C^{-1} + CA_2 C^{-1}
\end{pmatrix}
\]

This implies that:

\[
x^* = \left[ I_{2n} - \beta (A \otimes C) (D_T \otimes D_G) (A^{-1} \otimes C^{-1}) \right]^{-1} \hat{\alpha}
\]

\[
= (A \otimes C) \sum_{k \geq 0} \beta^k (D_T \otimes D_G)^k (A \otimes C)^{-1} \hat{\alpha}
\]

\[
= \left( \frac{1 - \gamma_h}{2 - \gamma_l - \gamma_h} \right) \begin{pmatrix}
CA_1 C^{-1} + \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) CA_2 C^{-1} & \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) CA_1 C^{-1} - \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) CA_2 C^{-1} \\
CA_1 C^{-1} - CA_2 C^{-1} & \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) CA_1 C^{-1} + CA_2 C^{-1}
\end{pmatrix} \begin{pmatrix}
\hat{\alpha}_l 1_n \\
\hat{\alpha}_h 1_n
\end{pmatrix}
\]

\[
\times \begin{pmatrix}
\hat{\alpha}_l [b(\beta, G) + \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) b((\gamma_h + \gamma_l - 1) \beta, G)] + \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) \hat{\alpha}_h [b(\beta, G) - b((\gamma_h + \gamma_l - 1) \beta, G)] \\
\hat{\alpha}_l [b(\beta, G) - b((\gamma_h + \gamma_l - 1) \beta, G)] + \left( \frac{1 - \gamma_l}{1 - \gamma_h} \right) \hat{\alpha}_h b(\beta, G) + \hat{\alpha}_h b((\gamma_h + \gamma_l - 1) \beta, G)
\end{pmatrix}
\]

The last equality is obtained by observing that:

\[
b(\beta, G) = \left( \sum_{k=0}^{+\infty} \beta^k G^k \right) 1_n
\]

\[
= \left( \sum_{k=0}^{+\infty} \beta^k (CD_G C^{-1})^k \right) 1_n
\]

\[
= C \left( \sum_{k=0}^{+\infty} \beta^k D_G^k \right) C^{-1} 1_n
\]

\[
= CA_1 C^{-1} 1_n
\]
By rearranging the terms, we obtain the equilibrium values given in the Proposition.

**Proof of Theorem 1:** The proof is relatively similar to that of Proposition 2. Indeed, $\Gamma$ can be diagonalized as:

$$\Gamma = AD_T A^{-1}, \text{ where } D_T = \begin{pmatrix} 
\lambda_1(\Gamma) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_T(\Gamma) 
\end{pmatrix}$$

In this formulation, $A$ is a $T \times T$ matrix where each $i$th column is formed by the eigenvector corresponding to the $i$th eigenvalue. Let us have the following notations:

$$A = \begin{pmatrix} 
 a_{11} & \cdots & a_{1T} \\
 \vdots & \ddots & \vdots \\
 a_{T1} & \cdots & a_{TT} 
\end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} 
 a_{11}^{(-1)} & \cdots & a_{1T}^{(-1)} \\
 \vdots & \ddots & \vdots \\
 a_{T1}^{(-1)} & \cdots & a_{TT}^{(-1)} 
\end{pmatrix}$$

where $a_{ij}^{(-1)}$ is the $(i, j)$ cell of the matrix $A^{-1}$.

The network adjacency matrix $G$ is symmetric and, therefore, it can also be diagonalized as:

$$G = CD_GC^{-1}, \text{ where } D_G = \begin{pmatrix} 
\lambda_1(G) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_n(G) 
\end{pmatrix}.$$

We can make use of the Kronecker product to rewrite the equilibrium system of the Bayesian game as:
\[
\begin{pmatrix}
  x(1) \\
  \vdots \\
  x(T)
\end{pmatrix}
= (I_{Tn} - \beta \Gamma \otimes G)^{-1}
\begin{pmatrix}
  \hat{\alpha}_1 I_n \\
  \vdots \\
  \hat{\alpha}_T I_n
\end{pmatrix}
\]

Applying the properties of the Kronecker product, this system becomes

\[
\begin{pmatrix}
  x(1) \\
  \vdots \\
  x(T)
\end{pmatrix}
= \left[ I_{Tn} - \beta (A \otimes C) (D_\Gamma \otimes D_G) (A^{-1} \otimes C^{-1}) \right]^{-1}
\begin{pmatrix}
  \hat{\alpha}_1 I_n \\
  \vdots \\
  \hat{\alpha}_T I_n
\end{pmatrix}
\]

First, let us show that \( I_{Tn} - \beta \Gamma \otimes G \) is non-singular. This is true if \( \beta \lambda_{\max}(\Gamma \otimes G) < 1 \). Since \( \Gamma \) is stochastic we have that \( \lambda_{\max}(\Gamma) = 1 \) and this condition boils down to \( 0 < \beta < 1/\lambda_{\max}(G) \). This shows that the system above has a unique solution and thus there exists a unique Nash-Bayesian equilibrium. This solution is interior since we have assumed that \( \alpha_1 > 0, \ldots, \alpha_M > 0 \), which implies that \( \hat{\alpha}_1 > 0, \ldots, \hat{\alpha}_T > 0 \).

Second, let us show that the equilibrium effort of agent \( i \) is a linear function of the Katz-Bonacich centrality measures. We have

\[
\left[ I_{Tn} - \beta (A \otimes C) (D_\Gamma \otimes D_G) (A^{-1} \otimes C^{-1}) \right]^{-1}
= (A \otimes C) \left( \sum_{k=0}^{\infty} \beta^k \begin{pmatrix} D_\Gamma \otimes D_G \end{pmatrix}^k \right) (A^{-1} \otimes C^{-1})
\]

We also have that \( (A \otimes C) \), \( (D_\Gamma \otimes D_G)^k \) and \( (A^{-1} \otimes C^{-1}) \) are each a \( Tn \times Tn \) matrix with

\[
A \otimes C =
\begin{pmatrix}
  a_{11} C & \cdots & a_{1T} C \\
  \vdots & \ddots & \vdots \\
  a_{T1} C & \cdots & a_{TT} C
\end{pmatrix}
\]

\[
\sum_{k \geq 0} \beta^k (D_\Gamma \otimes D_G)^k =
\begin{pmatrix}
  \sum_{k \geq 0} \beta^k \lambda_1^k (\Gamma) D_G^k & 0 & \cdots & 0 \\
  0 & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \cdots & 0 & \sum_{k \geq 0} \beta^k \lambda_T^k (\Gamma) D_G^k
\end{pmatrix}
\]

and

\[
A^{-1} \otimes C^{-1} =
\begin{pmatrix}
  a_{11}^{-1} C^{-1} & \cdots & a_{1T}^{-1} C^{-1} \\
  \vdots & \ddots & \vdots \\
  a_{T1}^{-1} C^{-1} & \cdots & a_{TT}^{-1} C^{-1}
\end{pmatrix}
\]
It is easily verified that
\[
\sum_{k \geq 0} (A \otimes C) \beta^k (D_T \otimes D_G)^k (A^{-1} \otimes C^{-1})
\]
\[
\begin{pmatrix}
\sum_{t=1}^{T} a_{tt} a_{t1}^{-1} A (\lambda_t (\Gamma) \beta, G) & \cdots & \cdots & \sum_{t=1}^{T} a_{tt} a_{tT}^{-1} A (\lambda_t (\Gamma) \beta, G) \\
\vdots & \cdots & \ddots & \vdots \\
\vdots & \cdots & \ddots & \vdots \\
\sum_{t=1}^{T} a_{TT} a_{T1}^{-1} A (\lambda_t (\Gamma) \beta, G) & \cdots & \cdots & \sum_{t=1}^{T} a_{TT} a_{tT}^{-1} A (\lambda_t (\Gamma) \beta, G)
\end{pmatrix}
\]
where
\[
A (\lambda_t (\Gamma) \beta, G) = C \sum_{k \geq 0} \beta^k \lambda_t^k (\Gamma) D_G^k C^{-1}
\]
is an \(n \times n\) matrix. Observe that
\[
A (\lambda_t (\Gamma) \beta, G) 1_n = b (\lambda_t (\Gamma) \beta, G)
\]
is a \(n \times 1\) vector of Katz-Bonacich centrality measures. For \(t = 1, ..., T\), define:
\[
\hat{\alpha}_t = \mathbb{E}_t [\alpha | \{ s_i = t \}] = \sum_{m=1}^{M} \mathbb{P} (\{ \alpha = \alpha_m \} | \{ s_i = t \}) \alpha_m
\]
Then, the result of the product
\[
[I_{Tn} - \beta (A \otimes C) (D_T \otimes D_G) (A^{-1} \otimes C^{-1})]^{-1} \begin{pmatrix}
\hat{\alpha}_1 1_n \\
\vdots \\
\hat{\alpha}_T 1_n
\end{pmatrix}
\]
depends on products of the form
\[
\hat{\alpha}_t \left( \sum_{t=1}^{T} a_{tt} a_{tj}^{-1} C (\lambda_t (\Gamma) \beta)^k D_G^k C^{-1} \right) 1_{Tn} = \hat{\alpha}_t \sum_{t=1}^{T} a_{tt} a_{tj}^{-1} \left[ C (\lambda_t (\Gamma) \beta)^k D_G^k C^{-1} 1_{Tn} \right]
\]
\[
= \hat{\alpha}_t \sum_{t=1}^{T} a_{tt} a_{tj}^{-1} b (\lambda_t (\Gamma) \beta, G),
\]
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which is a $Tn \times 1$ vector with entries equal to the combinations of the Katz-Bonacich centrality measures $b_t (\lambda_t (\Gamma) \beta, G)$, $t = 1, \ldots, T$. In other words, the equilibrium effort of each agent is:

$$x^*(1) \quad \ldots \quad x^*(\tau) \quad \ldots \quad x^*(T) = \begin{pmatrix}
\sum_{t=1}^{T} \alpha_t^{-1} a_{1t}^{-1} b_i (\lambda_t (\Gamma) \beta, G) + \ldots + \alpha_T^{-1} a_{Tt}^{-1} b_i (\lambda_t (\Gamma) \beta, G) \\
\vdots \\
\sum_{t=1}^{T} a_{rt}^{-1} b_i (\lambda_t (\Gamma) \beta, G) + \ldots + \sum_{t=1}^{T} a_{Tr}^{-1} b_i (\lambda_t (\Gamma) \beta, G)
\end{pmatrix} = \alpha \sum_{t=1}^{T} a_{rt}^{-1} b_i (\lambda_t (\Gamma) \beta, G) + \ldots + \alpha_T^{-1} a_{Tr}^{-1} b_i (\lambda_t (\Gamma) \beta, G)
$$

This means, in particular, that if individual $i$ receives a signal $s_i (\tau)$, then her effort is given by:

$$x_i^*(\tau) = \alpha \sum_{t=1}^{T} a_{rt}^{-1} b_i (\lambda_t (\Gamma) \beta, G) + \ldots + \alpha_T^{-1} a_{Tr}^{-1} b_i (\lambda_t (\Gamma) \beta, G) \quad (64)$$

Hence, we obtain the final expressions of equilibrium efforts as linear functions of own Katz-Bonacich centrality measures.

**Proof of Theorem 4:**

First, let us prove that this system has a unique solution and therefore there exists a unique Bayesian Nash equilibrium of this game. The system has one and only one well defined and non-negative solution if and only if $\beta_{\max} < 1/\lambda_{\max} (\Gamma \otimes G)$. Since $\lambda_{\max} (\Gamma \otimes G) = \lambda_{\max} (\tilde{\Gamma})$, the result follows.

Let us now characterize the equilibrium. Assumptions 1 and 4 guarantee that $\tilde{\Gamma}$ is well-defined, symmetric and thus diagonalizable. As a result, the proof is analogous to the case with incomplete information on $\alpha$ (see proof of Theorem 1).

**Proof of Remark 2:** Define a new matrix $\tilde{\Gamma} = (\tilde{\gamma}_{tr})_{t,r}$ as follows:

$$\tilde{\gamma}_{tr} = \sum_{m=1}^{M} \mathbb{P}(\{\beta = \beta_m\} \cap \{s_i = \tau\} \mid \{s_j = t\})$$

By definition we have that $\tilde{\gamma}_{tr} < \gamma_{tr}$ for all $t, \tau$ and $\sum_{\tau=1}^{T} \tilde{\gamma}_{tr} = 1$ for all $t$. This means that $0 \leq \tilde{\Gamma} \leq \bar{\Gamma}$ (elementwise inequalities). This implies that $\lambda_{\max} (\tilde{\Gamma}) \leq \lambda_{\max} (\bar{\Gamma}) = 1$. Thus, a sufficient condition for $I_{Tn} - \beta_{\max} \tilde{\Gamma} \otimes G$ to be non-singular is $\beta_{\max} < 1/\lambda_{\max} (G)$, and the result follows.