Mathematics III exam. Stockholm Doctoral Program. January 12, 2015

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Instructions Clearly state all steps towards the answer. Showing understanding of a working method is more important than getting all the algebra exactly correct. Calculators not capable of solving differential and/or difference equations are allowed. You may use a "cheat sheet" consisting of hand-written notes on one sheet of A4 paper (single or double-sided). No other aid is allowed.

There is no guarantee against the existence of typos or ambiguities in the questions. If you believe there is a typo or some missing information in a question, state your additional assumptions and interpretations clearly.

If you get stuck on a question, try to provide some arguments for how the problem should be solved and then go on to the other questions. It is also a good idea to read the whole exam before you start.

Your final grade will based on your performance in the exam (0-90 points) and in the homeworks (0-10 points). To pass the course you need a minimum of 50 points in total.

Good luck!

1. [20 points] Consider the difference equation

$$x_{t+2} - 5x_{t+1} + 6x_t = 4^t + t^2 + 3 \tag{1}$$

   (a) Find the general solution of the *homogeneous version* of equation (1). [5 points]

   (b) Use your result from (a) to find the general solution of (1). [10 points]

   (c) Is equation (1) globally asymptotically stable? Are there stationary states? If yes, are they globally asymptotically stable? [5 points]

2. [15 points] Consider the following system of first-order difference equations (written in matrix form):

$$X_{t+1} = AX_t, \text{ where } A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

   (a) Diagonalize the matrix A to find a simple formulation of the general solution. [10 points]

   (b) Find the general solution of the following nonhomogeneous equation [5 points]:

$$X_{t+1} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} X_t + \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

3. [15 points] **(Hard!)** Consider a standard autonomous first-order difference equation: $x_{t+1} = F(x_t)$. Suppose the equation has a stationary state x^*. Is it possible that x^* is unstable and globally asymptotically stable at the same time? If yes, provide an example. If no, provide a proof.

4. [20 points] Consider the following dynamic optimization problem:

$$\max_{\{u_t\}_{t=0}^T} \sum_{t=0}^{T} x_t^2(1+u_t) \quad \text{subject to } x_{t+1} = x_t(1-u_t),\ u_t \in [0,1],$$

with x_0 given.

(a) Find the value functions $J_T(x)$ and $J_{T-1}(x)$ and the corresponding optimal controls $u_T^*(x)$ and $u_{T-1}^*(x)$. [10 points]

(b) Show by induction that $J_{T-n}(x) = (n+2)x^2$ for $n = 0, 1, 2, \ldots, T$. Find the sequence of states and optimal controls $\{x_t^*, u_t^*\}_{t=0}^{T}$. [10 points]

5. [20 points] Let T, r, δ, x_0, x_T be positive constants. Consider the control problem

$$\max_{u(t)\in\mathbb{R}} \int_0^T -(x(t) - u(t) + 2)^2 e^{-rt} dt$$
$$\text{s.t.} \quad \dot{x}(t) = u(t) - \delta x(t), \;\; x(0) = x_0, \;\; x(T) = x_T$$

(a) Write down the conditions of the maximum principle. [5 points]

(b) Show that the Hamiltonian function is concave in (x, u). [5 points]

(c) Solve the problem with $T = 10$, $r = 0.1$, $\delta = 0.5$, $x_0 = 0$, $x_{10} = 8$. [10 points]