Competitive pricing strategies in social networks

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Abstract

We study pricing strategies of competing firms who sell heterogeneous products to a group of customers in a social network. Goods are substitutes and each customer gains network externalities from her neighbors who consume the same products. We show that there is a unique subgame-perfect equilibrium where, first, individuals decide their consumption of the goods and, second, firms choose the prices of each good for each consumer. We also fully characterize the equilibrium prices for any network structure, and relate these equilibrium outcomes to the familiar Katz-Bonacich network centrality measures. Contrary to the monopoly case, the equilibrium price of a customer not only depends on her own characteristics but also on others’ characteristics. We show that firms price discriminate and charge lower prices to more central consumers. This means that more central consumers obtain a larger discount because of their impact in terms of consumption on their neighbors. We also show that the firms’ equilibrium profits can decrease when either the network becomes denser or network effects are higher.

Keywords: social networks, pricing, competition, differentiated products

JEL classifications: L14, L13, D43, D85

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1 Introduction

The past decade has witnessed an emerging role of social networks in shaping individual choices. Customers now make consumption decisions largely based on whether their close friends, neighbors, and celebrities also adopt the same products. In economic terminology, these social goods have network externalities that connect individual benefits with others’ consumption/purchasing decisions.

Various sources of network externalities have been documented in economics. For telecommunication devices, these externalities are generated via physical connections. For operating systems and software packages, the externalities come from compatibility concerns. Network externalities may also arise purely from psychological reasons since individuals want to be perceived as part of their peers. The successes of Facebook, Linkedin, Twitter, Whatsapp, and various online game producers have confirmed the high potential of market profitability in social networking business.

These social interactions create abundant opportunities for service providers and thereby lead to intense competition among them. For instance, in the mobile and data services market, customers can choose among AT&T, Verizon, T-mobile, and other operators in the United States. In the market for mobile messaging apps, Whatsapp, Line, and WeChat are fighting for market shares. In such a scenario, customers are confronted with various options to stay connected with their friends, and service providers strive to induce the customers to lean towards their own products rather than their competitors’ products.

To better understand these issues, we provide a framework that examines product competition between firms when products are differentiated and exhibit local network effects. Each customer (player) has access to two products, which are offered by two firms. Within each product there are local network externalities amongst the customers in terms of their consumption utilities: a customer pays more attention to her close friends’ decisions than others’ choices. As in Ballester et al. (2006) and Zhou and Chen (2015), we develop a model with strategic complementarities in consumption choices so that more consumption from a customer reinforces other customers’ decisions to consume the same good. The firms incur heterogeneous costs of serving different customers and are allowed to charge discriminatory prices to these customers.

We show that there is a unique subgame-perfect equilibrium where, first, individuals decide their consumption of the goods and, second, firms choose the prices of each good for each consumer. We also provide a full characterization of the equilibrium prices, which can be decomposed into two parts. The first term corresponds to the monopoly price, which is independent of the network configuration. The second term is proportional to the Katz-Bonacich centrality measure of a customer (Bonacich 1987, Katz 1953). Thus, contrary to the monopoly case, the equilibrium prices exhibit strong network dependence. For example, in a complete graph, the price for a customer with a larger intrinsic value is higher when the marginal costs are constant across customers and across firms. For a bipartite graph (where there is a natural separation between two groups of customers), the customers in a smaller group receive a price discount. This holds true even if

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1 Network externalities exist if the value of a product increases when there are more users joining the network. Here, we are focusing on local network externalities where the value of consuming a product for a given agent increases when others directly connected to this agent consume this product.
intrinsic values and marginal costs are the same across all customers and firms. As a result, the knowledge regarding the network structure becomes crucial for the profit maximization.

There are, however, examples in which both firms optimally charge uniform prices when certain assumptions about the network configuration and the intrinsic values are satisfied. For instance, in a regular graph (i.e., all customers have the same degree), we show that, in equilibrium, both firms adopt uniform pricing when intrinsic values and marginal costs are the same for all customers and firms. Thus, a common price is offered to all customers in the network, irrespective of their network positions. This uniform pricing strategy is particularly desirable from the practical standpoint, if it happens that the customers’ relationships are described by such a network configuration.

In addition, we show that enhancing network externalities among customers actually pushes the equilibrium price downwards. This is because it intensifies price competition. Such a phenomenon can never arise in a monopoly setting. We also document the possibility that firms’ equilibrium profits can be reduced when either the network gets denser or the network effect is strengthened. By contrast, a monopoly firm always collects a higher profit under the same circumstances. Therefore, competition can lead to substantially different implications of the pricing strategies as well as firms’ profitability. Valuable/successful marketing strategies such as adding more links among customers or increasing network effects in the monopoly case should be reconsidered with caution when competition is introduced.

Finally, We extend our analysis to accommodate the oligopoly competition of more than two firms and show that the results are robust and can therefore be generalized. We also consider the case of asymmetric firms and cross-network effects.

The remainder of this paper unfolds as follows. Section 2 reviews some relevant literature. Section 3 introduces the model setup. Section 4 examines the equilibrium outcomes in the customers’ consumption stage. Section 5 characterizes the firms’ equilibrium pricing strategies. Section 6 discusses the firms’ equilibrium profits. Several extensions are included in Section 7. We make some concluding remarks in Section 8. Appendix A provides some matrix notations used throughout this paper and the definition of the Katz-Bonacich centrality. Appendix B gives all the proofs of our technical results. Appendix C considers the case of a single representative customer when there is no network effects. Finally, Appendix D provides additional results for the oligopoly model.

## 2 Literature review

A large literature in economics has investigated the issue of network effects/network externalities. The classical papers primarily focus on the aggregate level of network externalities (e.g., Rohlfs (1974), Katz and Shapiro (1985), Farrell and Saloner (1986)). Monopoly pricing of network goods is modeled and analyzed in various papers such as Cabral et al. (1999), Dybvig and Spatt (1983), and Ochs and Park (2010). \(^2\) The competitive pricing problem is mostly studied in the context of two-

\(^2\)See Economides (1996) for an extensive survey of this literature.
sided networks in which players on one side care about the aggregate contributions of those on the other side (see, e.g., Armstrong (2006), Caillaud and Jullien (2003), and Rochet and Tirole (2006)). Using our terminology, this corresponds to the complete bipartite network case. In contrast, we study local network effects and explicitly model the network structure among players.

Ballester et al. (2006) provide a tractable approach to study network games with strategic complementarities amongst players using linear-quadratic utility functions. They show the Nash equilibrium in effort is proportional to the “Katz-Bonacich centrality” of each agent. Two recent contributions by Bloch and Quéré (2013) and Candogan et al. (2012) incorporate the pricing decisions into the framework of Ballester et al. (2006). They independently show that if the monopoly firm can price discriminate among players and the network effects are symmetric (undirected), the resulting optimal prices do not take into account the network structure. Galeotti and Fainmesser (2015) use a somewhat different framework and consider the possibility that the firm knows partial information about the players’ in-degrees or/and out-degrees (i.e., how influential they are or how significantly they are influenced by their neighbors). Allowing for third-degree price discrimination, Galeotti and Fainmesser (2015) show that the optimal pricing depends on the network configuration as well as the firm’s knowledge about it.

Our work is related to Ballester et al. (2006), Bloch and Quéré (2013), Candogan et al. (2012), and Galeotti and Fainmesser (2015). However, contrary to Ballester et al. (2006), we introduce multiple products framework. As a result, the consumption game amongst players is not only driven by network externalities but also by the interdependence between different products. Moreover, we extend Bloch and Quéré (2013), Candogan et al. (2012), and Galeotti and Fainmesser (2015) by allowing price competition between firms. We show that the competitive pricing strategies can in general exhibit strong network dependence and the firms’ profitability can be reduced when the network becomes denser. When the products are independent, our model degenerates to the models of Bloch and Quéré (2013) and Candogan et al. (2012). We show that the dependence of prices on network structure would not arise in this knife-edge case. While the network-dependent pricing shares some similarities with that in Galeotti and Fainmesser (2015), the underlying drivers are very different. In our model, the firms possess perfect knowledge about the network configuration; thus, the sole driver for network-dependent pricing is product competition. In contrast, in Galeotti and Fainmesser (2015) it is the firm’s limited knowledge that creates such dependence.

In a related paper, Aoyagi (2014) assumes that both products are incompatible and, therefore, each customer only consumes at most one product from two firms. With this extreme form of product substitution, the author focuses on the condition under which both firms charging marginal costs is a subgame perfect equilibrium. He also illustrates how multiplicity of equilibria in the consumption stage arises and provides some equilibrium selection criteria. In contrast, in our paper, the consumption behavior is continuous and the Nash equilibrium in the consumption stage

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*The economics of networks is a growing field. For recent overviews, see Jackson (2008), Ioannides (2012), Jackson et al. (2015) and Jackson and Zenou (2015). See also Shi (2008), Deroian and Gannon (2006), Billand et al. (2014), Carroni and Righi (2015), Currarini and Feri (2015), Bimpikis et al. (2015) and Ushchev and Zenou (2015).*
is shown to be unique. Moreover, we allow for any degree of substitution (or complements) between products, and generate closed-form expressions on players’ consumptions, equilibrium prices, and firms’ profits. These analytical expressions allow us to conduct various comparative statics results on the network structure, degree of substitution, and network effects.

Banerji and Dutta (2009) also consider price competition in the presence of local network effects. However, they focus on the uniform pricing case in which the firms cannot price discriminate between players. Similar assumptions on the flat/uniform pricing are imposed in Armstrong (2006), Caillaud and Jullien (2003), and Rochet and Tirole (2006) as well. As in Aoyagi (2014), they provide conditions under which either a firm monopolizes the market or the market is completely segmented. Ambrus and Argenziano (2009) and Jullien (2011) discuss the pricing competition among platforms in two-sided markets. Thus, the underlying network structure is bipartite. Fazeli and Jadababaie (2012) consider a similar problem too. However, they assume that the firms charge uniform pricing to the customers. Moreover, in their model each customer has an inelastic unit demand and decides the fraction of consumptions between the two products.

3 Model

We consider a duopoly model in which two firms intend to sell products to n customers/players, indexed by \( i = 1, 2, \ldots, n \), who are embedded in a social network. Let \( N = \{1, 2, \ldots, n\} \) denote the set of customers. The network with n players is described by the \( n \times n \) adjacency matrix \( G = (g_{ij}) \). We assume that there are no self loops, i.e. \( g_{ii} = 0 \) and that \( g_{ij} = g_{ji} \), i.e., \( G \) is symmetric. Each customer consumes two products, A and B, which are offered by firms A and B. Firm A(B) is the sole provider for product A(B); thus, they have full price discrimination power over the products they offer. However, the two products are interdependent, and there are network externalities amongst the customers in terms of their consumption utilities. Throughout the paper, we will use customers and players interchangeably.

**Customers’ utilities.** Customer \( i \)’s utility function is expressed as follows:

\[
 u_i = a_i^A x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2} (x_i^A)^2 + \frac{1}{2} (x_i^B)^2 + \beta x_i^A x_i^B \right\} \\
+ \delta \sum_{j=1}^{n} g_{ij} x_i^A x_j^A + \delta \sum_{j=1}^{n} g_{ij} x_i^B x_j^B \\
- p_i^A x_i^A - p_i^B x_i^B.
\]

This utility consists of three parts.

The first part of (1), \( a_i^A x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2} (x_i^A)^2 + \frac{1}{2} (x_i^B)^2 + \beta x_i^A x_i^B \right\} \), corresponds to customer \( i \)’s own consumption utility and is expressed as a cost-benefit function. Parameter \( a_t^i \) measures the intrinsic marginal utility of customer \( i \) for product \( t = A, B \). The quadratic terms capture the decreasing marginal returns from consuming each product, and the cross term \( \beta x_i^A x_i^B \) depicts the
interconnection between products. While our analysis is applicable to general values of \( \beta \in (-1, 1) \),
we will restrict our discussions to the special case when \( \beta \in [0, 1) \). When \( \beta \) is close to 1, the
products are almost perfectly substitutable. When \( \beta = 0 \), the two products are independent. As a
result, \( \beta \) measures the degree of substitution between the two goods or equivalently the degree of
differentiation between goods \( A \) and \( B \). Indeed, the higher is \( \beta \), the more substitute the goods are
and, thus, the less differentiated they are.

The second part of (1), \( \delta \sum_{j=1}^{n} g_{ij} x_{i}^{A} x_{j}^{A} + \delta \sum_{j=1}^{n} g_{ij} x_{i}^{B} x_{j}^{B} \), captures the network externalities
within each product. The intensity of the network effects is described by \( \delta \geq 0 \) since a higher
\( \delta \) indicates that a customer’s utility depends more heavily on others’ decisions. We assume that
\( g_{ij} \geq 0 \) so that a higher consumption of product \( t = A, B \) from customer \( j \) linked to \( i \) increases the
marginal utility of consuming product \( t = A, B \) for consumer \( i \).

The third term \(- p_{i}^{A} x_{i}^{A} - p_{i}^{B} x_{i}^{B} \) is the total expenditure on both products for consumer \( i \),
where \( p_{i}^{A} \) and \( p_{i}^{B} \) are the prices charged by firms \( A \) and \( B \) to consumer \( i \), respectively. The marginal cost
of producing product \( t \) for customer \( i \) is \( c_{t}^{i} \), where \( i = 1, \cdots, n \) and \( t = A, B \). Naturally, we assume
\( a_{i}^{A} > c_{i}^{A} \) and \( a_{i}^{B} > c_{i}^{B} \), \( \forall i \). Otherwise, the values of these products would not justify the production
costs.

**Two benchmark models.** Let us now discuss two benchmark models in the literature. The
**first benchmark model** is when \( \delta = 0 \) (no network effects) so that customer \( i \) ’s utility function
reduces to

\[
    u_{i} = a_{i}^{A} x_{i}^{A} + a_{i}^{B} x_{i}^{B} - \left\{ \frac{1}{2} (x_{i}^{A})^{2} + \frac{1}{2} (x_{i}^{B})^{2} + \beta x_{i}^{A} x_{i}^{B} \right\} - p_{i}^{A} x_{i}^{A} - p_{i}^{B} x_{i}^{B}.
\]  

(2)

The above setup is commonly used in the industrial organization literature (Dixit 1979; Singh and
Vives 1984), especially in love-for-variety models of monopolistic competition with linear-quadratic
utility (see e.g. Ottaviano et al. (2002); Melitz (2003); Combes et al. (2008), where \( \beta \) serves as
an (inverse) measure of product differentiation. Indeed, since the network linkages disappear, the
model degenerates to the case of \( n \) independent representative customers and the utility function
corresponds to a standard linear-quadratic model with product differentiation for the case of two
goods and \( n \) consumers. As a result, the two firms engage in a price competition to attract
each representative customer. In Appendix [C] we provide the detailed derivations for a single
representative customer where the utility function is given by (2).

The **second benchmark model** is when \( \beta = 0 \) so that customer \( i \) ’s utility is now equal to

\[
    u_{i} = \left\{ a_{i}^{A} x_{i}^{A} - \frac{1}{2} (x_{i}^{A})^{2} + \frac{1}{2} (x_{i}^{B})^{2} + \delta \sum_{j=1}^{n} g_{ij} x_{i}^{A} x_{j}^{A} - p_{i}^{A} x_{i}^{A} \right\} + \left\{ a_{i}^{B} x_{i}^{B} - \frac{1}{2} (x_{i}^{B})^{2} + \delta \sum_{j=1}^{n} g_{ij} x_{i}^{B} x_{j}^{B} - p_{i}^{B} x_{i}^{B} \right\}.
\]
Because the two terms in the big brackets are totally separable, there is no interaction between the two products. Consequently, each firm simply acts as a monopoly in their own market, and the problem within each market degenerates to the single-product setting studied by [Bloch and Quérôu (2013)] and [Candogan et al. (2012)].

Assumptions. We now introduce some assumptions regarding the degree of activity interdependence and the intensity of network effects. Denote by $\lambda_1(G)$ the largest eigenvalue of the adjacency matrix $G$.

**Assumption 1.** $1 - |\beta| - \delta \lambda_1(G) > 0$.

Assumption 1 guarantees the existence and boundedness of the equilibrium. When $\beta = 0$, this condition is equivalent to $\delta < 1/\lambda_1(G)$, which is commonly assumed in the network literature with single activities ([Ballester et al. (2006)] and [Zhou and Chen (2015)]).

The next two assumptions concern the intrinsic marginal utilities and marginal costs. Even if the profit of the firms will be introduced later, for $t = A, B$, let us denote by $c_t^i$ the marginal of producing good $t$ for consumer $i$. To simplify exposition, in some parts of the paper, we will assume the following semi-symmetry (SS) assumption:

**Assumption 2.** $a_A^i = a_B^i = a_i, c_A^i = c_B^i = c_i, i = 1, 2, \cdots, n$.

We call Assumption 2 semi-symmetry (SS) because it implies that initially, for each customer $i$, products $A$ and $B$ are symmetric without network effects. Therefore, there is no endowed biases towards either product. In some examples, we further impose symmetry on the customers’ intrinsic marginal utilities.

**Assumption 3.** $a_A^i = a_B^i = a, c_A^i = c_B^i = c, i = 1, 2, \cdots, n$.

We call this assumption full symmetry (FS), because it implies that all customers are the same except for their network positions. We should emphasize that the network matrix $G$ remains arbitrary (albeit symmetric) despite Assumption 3.

**Timing.** The sequence of events is as follows. First, firms $A$ and $B$ determine the price vectors $p^A = (p_1^A, \cdots, p_n^A)$ and $p^B = (p_1^B, \cdots, p_n^B)$ simultaneously. Second, after observing $p^A$ and $p^B$, each customer $i$ non-cooperatively chooses her consumption bundle $x_i = (x_i^A, x_i^B)$ and this choice is made simultaneously with all other consumers. The solution concept for this two-stage game is subgame-perfect Nash equilibrium (SPNE). By backward induction, we first derive the equilibrium outcomes amongst customers in the consumption stage in Section 4. We then analyze the firms’ pricing strategies in Section 5.

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5See Appendix A for all matrix and vector notations and some related definitions.
4 Consumption Stage

We first take as given the price vectors $p^A$ and $p^B$ offered by firms A and B and solve the customers’ equilibrium consumptions. Define the following two $n \times n$ matrices:

$$M^+ := [(1 + \beta)I_n - \delta G]^{-1}, \quad M^- := [(1 - \beta)I_n - \delta G]^{-1}. \tag{3}$$

For $t = A, B$, denote $x^t = (x^t_1, \ldots, x^t_n)'$, $a^t = (a^t_1, \ldots, a^t_n)'$ and $p^t = (p^t_1, \ldots, p^t_n)'$. We have the following theorem, which is directly adapted from Chen et al. (2015), after taking prices into account:

**Proposition 1.** If Assumption 1 holds, then, for any prices $p^A$ and $p^B$ charged by firms A and B, there exists a unique equilibrium in the consumption stage with

$$\begin{bmatrix} x^A \\ x^B \end{bmatrix} = \begin{bmatrix} M^+ + M^- & M^+ - M^- \\ M^+ - M^- & M^+ + M^- \end{bmatrix} \begin{bmatrix} a^A - p^A \\ a^B - p^B \end{bmatrix}. \tag{4}$$

or, equivalently,

$$\begin{cases} x^A = M^+ \left\{ \frac{a^A + a^B}{2} - \frac{p^A + p^B}{2} \right\} + M^- \left\{ \frac{a^A - a^B}{2} - \frac{p^A - p^B}{2} \right\} \\ x^B = M^+ \left\{ \frac{a^A + a^B}{2} - \frac{p^A + p^B}{2} \right\} - M^- \left\{ \frac{a^A - a^B}{2} - \frac{p^A - p^B}{2} \right\} \end{cases} \tag{4}$$

This proposition provides a full characterization of the equilibrium consumption of the two goods for any network structure. It gives a novel decomposition of the best-reply functions by showing that what determines the consumption of each good is the sum and difference of the marginal utility of consumption and prices of each product multiplied by two matrices capturing the different paths in the network. These matrices $M^+$ and $M^-$ are clearly related to the Katz-Bonacich centrality (which is defined in Appendix A), with a decay factor that not only depend on $\delta$, the intensity of network externalities between directly connected individuals for each product, but also on $\beta$, the degree of substitutability between the two products. Indeed, when consumers make their consumption decisions, they take into account both local network effects and the degree of substitution between the two products. We have here assumed a linear-quadratic utility function for the consumers, which allows us to have linear best responses and a clean characterization of the equilibrium consumptions. As showed by Bramoullé et al. (2014), our equilibrium analysis and results on optimal pricing carry over to a more general class of utility functions that induce linear best responses.

Let now impose a symmetry condition to get a cleaner expression.

**Corollary 1.** Suppose $a^A = a^B = a$ and $p^A = p^B = p$. If Assumption 1 holds, then, for any prices $p^A$ and $p^B$, there exists a unique equilibrium given by

$$x^A = x^B = M^+(a - p).$$
We can also obtain some comparative statics results. Because \( \beta \geq 0 \) and \( M^+ \preceq M^- \), the following results are immediate to obtain:

\[
\frac{\partial x^A}{\partial a^A} = \frac{-x^A}{\partial p^A} = \frac{M^+ + M^-}{2} > 0, \quad \text{and} \quad \frac{\partial x^B}{\partial a^A} = \frac{-x^B}{\partial p^A} = \frac{M^+ - M^-}{2} \leq 0. \quad (5)
\]

Therefore, when \( p^A(p^B) \), the price of product \( A \) (\( B \)), increases, each customer will consume less of product \( A \) (\( B \)) and more of product \( B \) (\( A \)) (pure substitution effect). This result holds for all networks \( G \) and parameters \( \delta, \beta \). These comparative statics formulas have implications for the equilibrium prices determined in the next section.

5 Pricing Stage

In Proposition \[\text{Proposition 1}\] we studied the equilibrium consumption bundles for all customers in the social network given the fixed prices charged by firms \( A \) and \( B \). In this section, we solve the first stage of the game, i.e. the pricing decisions of both firms. For \( t = A, B \), denote \( c^t = (c^t_1, \ldots, c^t_n)' \). Then, the equilibrium total profit for firm \( A \) can be expressed as:

\[
\Pi_A(p) = \langle p^A - c^A, x^A(p^A, p^B) \rangle
= \langle p^A - c^A, \frac{M^+ + M^-}{2}(a^A - p^A) + \frac{M^+ - M^-}{2}(a^B - p^B) \rangle
\]

where the demand functions \( x^A(p^A, p^B) \) are given in Proposition \[\text{Proposition 1}\]. Similarly, the profit of firm \( B \) is given by

\[
\Pi_B(p) = \langle p^B - c^B, x^B(p^A, p^B) \rangle
= \langle p^A - c^A, \frac{M^+ - M^-}{2}(a^A - p^A) + \frac{M^+ + M^-}{2}(a^B - p^B) \rangle.
\]

5.1 Collusive pricing

Before we study the competitive pricing, we first analyze the case where the two firms determine the prices \( p^A \) and \( p^B \) jointly to maximize their total profits. This could be the case when two firms merge into a single firm that controls both \( p^A \) and \( p^B \). In such a benchmark, the optimal \textit{collusive prices} solve

\[
\max_{\{p^A, p^B\}} \{\Pi_A(p) + \Pi_B(p)\}
\]

Let \( \bar{p}^A, \bar{p}^B \) be the solutions of this program.

\footnote{The inner product of two vectors \( x = (x_1, \ldots, x_n)' \) and \( y = (y_1, \ldots, y_n)' \) in \( \mathbb{R}^n \) is denoted as \( \langle x, y \rangle = \sum_i x_i y_i \).}
Lemma 1. Suppose that the two firms jointly determine their prices. If Assumption 1 holds, then there is a unique equilibrium and the corresponding collusive prices are given by

$$\bar{p}^A = \frac{a^A + c^A}{2}, \quad \bar{p}^B = \frac{a^B + c^B}{2}.$$

Lemma 1 shows that the collusive price for a customer $i = 1, ..., n$ consuming product $t = A, B$ only depends on this customer’s marginal utility and the marginal cost of product $t$. It is, however, independent of the network structure $G$, the strength of network effect $\delta$, and the degree of substitution $\beta$ between the two products. Moreover, if both firms are symmetric (i.e., Assumption 2 holds), then the collusive prices are also symmetric, and $\bar{p}^A = \bar{p}^B = \frac{a + c}{2}$. This network-independent result in the two-product case shares some similarity with the outcomes in the monopoly setting studied by [1] and Candogan et al. (2012), which have only one product. It also gives us a useful benchmark for the equilibrium prices with competition.

5.2 Competitive pricing

We now return to the competitive setting. First, we derive the best-reply function from an individual firm’s perspective. This partial equilibrium analysis is useful to obtain some intuition of the results. In certain scenarios, not all the firms have the flexibility to adjust the prices. For instance, a firm may have committed to offer the products for free (or at the marginal cost). In such a scenario, how should a price-setting firm charge the customers?

Suppose the price of the second firm $p^B$ is fixed and we are interested in firm $A$’s best response. From (6), let us differentiate $\Pi^A(p)$ with respect to $p^A$ and setting it to 0. We obtain:

$$\frac{M^+ + M^-}{2}(a^A - p^A) + \frac{M^+ - M^-}{2}(a^B - p^B) - \frac{M^+ + M^-}{2}(p^A - c^A) = 0,$$

or equivalently

$$(M^+ + M^-)(a^A + c^A - 2p^A) = -(M^+ - M^-)(a^B - p^B).$$

Therefore, the best response for firm $A$ is

$$\text{BR}^A(p^B) = \frac{a^A + c^A}{2} + \frac{1}{2}(M^+ + M^-)^{-1}(M^+ - M^-)(a^B - p^B) = \frac{a^A + c^A}{2} - \frac{1}{2}\beta[I - \delta G]^{-1}(a^B - p^B)$$

where the last step follows from the fact that

$$(M^+ + M^-)^{-1}(M^+ - M^-) = -\beta[I - \delta G]^{-1}.$$

In a similar way, we can obtain the best response of firm $B$:

$$\text{BR}^B(p^A) = \frac{a^B + c^B}{2} - \frac{1}{2}\beta[I - \delta G]^{-1}(a^A - p^A)$$
It is easily verified that both best-reply functions are monotone, i.e., $\text{BR}^A(p^B)$ is increasing in $p^B$ and $\text{BR}^B(p^A)$ is increasing in $p^A$.

The equilibrium prices are then pinned down by these two best-reply functions:

$$
\begin{align*}
    p^{*A} &= \text{BR}^A(p^{*B}) = \frac{a^A+c^A}{2} - \frac{1}{2} \beta [I - \delta G]^{-1}(a^B - p^{*B}) \\
    p^{*B} &= \text{BR}^B(p^{*A}) = \frac{a^B+c^B}{2} - \frac{1}{2} \beta [I - \delta G]^{-1}(a^A - p^{*A})
\end{align*}
$$

(8)

We now state our main result.

**Theorem 1.** Suppose Assumptions 1 and 2 hold. Then, there exists a unique equilibrium in the pricing stage in which both firms charge $p^{*A} = p^{*B} = p^*$:

$$
p^* = \frac{a + c}{2} - \frac{\beta}{2} [(2 - \beta)I_n - 2\delta G]^{-1}(a - c).
$$

(9)

First, let us show how we determine the equilibrium prices. By (8), the equilibrium prices must be symmetric when Assumption 2 holds. Suppose $p^{*A} = p^{*B} = p^*$ is a symmetric equilibrium. Therefore, $x^A = x^B = x^* = M^+(a - p^*)$ by Corollary 1 and $\Pi^A = \Pi^B = \langle x^*, p^* - c \rangle$. Now suppose firm A unilaterally deviates by decreasing her price vector by $\Delta p^{*A}$, i.e., $p^{*A} \sim p^* - \Delta p^{*A}$. This decrease has two effects. On the one hand, the price margins per unit are lower and the corresponding total marginal loss is $\langle x^*, \Delta p^{*A} \rangle = \langle M^+(a - p^*), \Delta p^{*A} \rangle$. On the other hand, there is a marginal benefit due to demand enhancing, which is $\langle \Delta x^A, p^* - c \rangle$. The change in consumption for product $A$ due to lower $p^{*A}$ is $\Delta x^A = \frac{M^+ + M^-}{2} \Delta p^{*A}$ by equation (5). In equilibrium, both effects must cancel out, i.e.,

$$
\langle M^+(a - p^*), \Delta p^{*A} \rangle = \langle \frac{M^+ + M^-}{2} \Delta p^{*A}, p^* - c \rangle,
$$

which holds for any price change $\Delta p^{*A} \in \mathbb{R}^n$. As a consequence, we obtain that

$$
M^+(a - p^*) = \frac{1}{2}(M^+ + M^-)(p^* - c).
$$

(10)

Plugging the values of $M^+$ and $M^-$ in this equation and simplifying yield the results stated in Theorem 1.

Second, let us give some economic intuition of the equilibrium price vector $p^*$ in Theorem 1. This equilibrium price is made of two terms. By Lemma 1, the first term $\frac{a + c}{2}$ is the collusive price (or the equilibrium price without competition or when goods are independent, i.e. $\beta = 0$) and it is also equal to the equilibrium price when $\beta = 0$, i.e. when products are independent. The second

---

7 Here we have implicitly used the fact that both $M^+$ and $M^-$ are symmetric, which is indeed the case since $G$ is symmetric.
(adjustment) term is due to competition. In particular, it is easily verified that

\[ \frac{\beta}{2} \left( (2 - \beta) I_n - 2\delta G \right)^{-1} (a - c) = \frac{\beta}{2(2 - \beta)} b \left( G, \frac{2\delta}{2 - \beta}, a - c \right) . \]

where \( b(G, \frac{2\delta}{2 - \beta}, a - c) \) is the vector of Katz-Bonacich centralities (defined in Appendix A.2) for network \( G \), discount factor \( \frac{2\delta}{2 - \beta} \) and weight \( a - c \). Because \( \beta > 0 \), this term is always positive, which means that competition always drives prices downwards. More importantly, this term is customer specific and it can be written as a Katz-Bonacich centrality measure. Because different customers obtain different price adjustments, depending on \( \beta, \delta \) and \( G \), in equilibrium, the firms do exercise price discrimination. Indeed, since the second (adjustment) term is proportional to the Katz-Bonacich centrality measure, the more central the consumers are in the network, the lower the price they pay for consuming the goods \( A \) and \( B \). This is a remarkable result, which shows that more central consumers obtain a larger discount because of their impact in terms of consumption on their neighbors. In other words, when firms have a precise information about “who influences whom” in the network, they can set different prices for different consumers in order to leverage network effects and earn larger profits. The most central agents receive each product at a lower price while others, less central, are exploited by the firms and pay a higher price. There are many examples of price discrimination based on the degree of influence of consumers. For example, in the case of fashion goods, firms offer free or very-low prices goods to selected influential consumers such as celebrities. Another example is Klout.com, a site that collects data about consumers from social networks to estimate their influential power. For example, Cathay Pacific offers lounge access to customers (of any airline) that have a high Klout score. Other companies such as online fashion retailers Bonobos and Gilt offer discounts, whereas Capital One provides increased credit card rewards for customers with high influence scores. Theorem 1 gives a precise definition of who are the influential consumers (defined by their Katz-Bonacich centrality) and how competing firms set prices that take into account this aspect.

It is easily verified that, when either \( \beta \) or \( \delta \) increases, the adjustment term increases and thus, with a higher degree of product substitution or stronger network effects, the equilibrium prices decrease.

We now apply Theorem 1 to the two benchmark cases presented in Section 3, i.e. when \( \beta = 0 \) and when \( \delta = 0 \). These special cases exhibit very peculiar properties.

**Corollary 2.** Suppose Assumptions 1 and 2 hold.

1. When \( \delta = 0 \) (no network effect), there is a unique equilibrium given by:

   \[ p^* = \frac{(1 - \beta) a + c}{2 - \beta} . \]

2. Knife-edge case: \( \beta = 0 \):
   - When \( \beta = 0 \) (two independent networks), \( p^* = \frac{a + c}{2} \), which is independent of parameter
\( \delta \) and network matrix \( \mathbf{G} \).

- For a fixed nonempty network \( \mathbf{G} \), if \( \mathbf{p}^* \) is independent of \( \delta \), then it must be the case that \( \beta = 0 \).
- For any pair of customers \( i, j \) with \( y_{ij} > 0 \), suppose \( \delta > 0 \) and customer \( i \)’s equilibrium price \( p_i \) is independent of \( a_j \) or \( c_j \). Then it must be the case that \( \beta = 0 \).

The first part of Corollary 2 shows that when \( \beta = 0 \) (the two products are independent), each firm behaves as a monopolist in her own product market. Therefore, the problem degenerates to the monopoly setup and the intensity of network effects \( \delta \) has no impact on the optimal (equilibrium) prices. This surprising result has been established by Bloch and Quérou (2013) and Candogan et al. (2012). This result is very different to the one obtained in Theorem 1 for the duopoly case where both the structure of the network and the intensity of network effects matter in the price determination of the goods. In other words, the competition between firms make the network effects non negligible. Corollary 2 also shows that this surprising result crucially depends on the product independence assumption or the monopoly assumption. In the presence of product competition, network does matter. In addition, it also provides the identification result in the reverse direction. Unless the products are independent (i.e., unless the two firms have no interactions), the equilibrium prices must utilize the knowledge of the network structure.

5.3 Comparative statics

Next, we derive some comparative statics results.

**Definition 1.** We say that \( \mathbf{G}' \succ \mathbf{G} \) if \( \mathbf{G}' \) contains \( \mathbf{G} \) as a subgraph and there are additional links in \( \mathbf{G}' \) but not in \( \mathbf{G} \).

This provides an incomplete ordering of graphs and we can compare two graphs if the sets of links are nested. Clearly, \( \mathbf{G}' \) is denser than \( \mathbf{G} \) and \( \lambda_1(\mathbf{G}') \geq \lambda_1(\mathbf{G}) \) if \( \mathbf{G}' \succ \mathbf{G} \). Let \( \text{id}_{\{i\}} \) denote the indicator function.

**Proposition 2.** Suppose Assumptions 7 and 3 hold.

1. We have:

\[
\frac{\partial p_i^*}{\partial a_j} = \frac{1}{2} \text{id}_{\{i=j\}} - \frac{\beta}{2(2-\beta)} m_{ij}(\mathbf{G}, \frac{2\delta}{2-\beta}) = \begin{cases} \frac{1}{2} - \frac{\beta}{2(2-\beta)} m_{ii}(\mathbf{G}, \frac{2\delta}{2-\beta}) & i = j; \\ - \frac{\beta}{2(2-\beta)} m_{ij}(\mathbf{G}, \frac{2\delta}{2-\beta}) & i \neq j; \end{cases}
\]

Moreover, \( 0 < \frac{\partial p_i^*}{\partial a_i} < \frac{1}{2}, \forall i \) and \( -\frac{1}{2} < \frac{\partial p_i^*}{\partial a_j} < 0 \) if \( i \neq j \).
2. We have:
\[
\frac{\partial p^*_i}{\partial c_j} = \frac{1}{2} \mathbf{1}_{(i=j)} + \frac{\beta}{2(2-\beta)} m_{ij}(\mathbf{G}, \frac{2\delta}{2-\beta}) \cdot \begin{cases} 
\frac{1}{2} + \frac{\beta}{2(2-\beta)} m_{ii}(\mathbf{G}, \frac{2\delta}{2-\beta}) & i = j; \\
\frac{2\beta}{2(2-\beta)} m_{ij}(\mathbf{G}, \frac{2\delta}{2-\beta}) & i \neq j; 
\end{cases}
\]

Moreover, \( \frac{\partial p^*_i}{\partial c_j} \geq 0 \) for all \( i, j \), and \( \frac{1}{2} < \frac{\partial p^*_i}{\partial c_i} < 1 \), \( \forall i, 0 < \frac{\partial p^*_i}{\partial c_j} < \frac{1}{2} \) if \( i \neq j \).

3. We have:
\[
\frac{\partial p^*}{\partial \delta} = -\beta [(2-\beta)\mathbf{I}_n - 2\delta \mathbf{G}]^{-2} \mathbf{G}(\mathbf{a} - \mathbf{c}) \leq 0,
\]
and thus \( \frac{\partial p^*_i}{\partial \delta} \leq 0 \) for any \( i \).

4. We have:
\[
\frac{\partial p^*}{\partial \beta} = -\frac{1}{2}[(2-\beta)\mathbf{I}_n - 2\delta \mathbf{G}]^{-1} (\mathbf{a} - \mathbf{c}) - \frac{\beta}{2} [(2-\beta)\mathbf{I}_n - 2\delta \mathbf{G}]^{-2} (\mathbf{a} - \mathbf{c}),
\]
and thus \( \frac{\partial p^*_i}{\partial \beta} \leq 0 \) for any \( i \).

5. If \( \mathbf{G}' \succ \mathbf{G} \), then \( p^*(\mathbf{G}') \preceq p^*(\mathbf{G}) \).

The comparative statics results can be best understood using the decomposition formula \( \mathbf{9} \). Recall that
\[
p^*_j = \frac{a_j + c_j}{2} - \frac{\beta}{2(2-\beta)} b_j \left( \mathbf{G}, \frac{2\delta}{2-\beta}, (\mathbf{a} - \mathbf{c}) \right).
\]

When \( a_j \) increases, it impacts \( p^*_j \) in two ways. The first term, \( \frac{a_j + c_j}{2} \), must increases. In addition, the second (adjustment) term also increases as the weight in the \( (\mathbf{a} - \mathbf{c}) \) in Katz- Bonacich centrality measures increases. However, the first effect dominates the second one and therefore the total effect on \( p_j \) is less than a half of the increase in \( a_i \). By contrast, for player \( i \neq j \), \( p^*_i \) simply drops as \( b_i \) increases for \( i \).

Similarly, when the cost \( c_j \) for player \( j \) increases, this player is charged with a higher price, i.e., \( p_j^* \) goes up. Moreover, the cost increase of a single player \( j \) passes through all other players \( i \neq j \) via their social interactions. Formally, the Katz- Bonacich centrality \( b_i \) not only depends on player \( i \)'s characteristic \( a_i - c_i \) but also on all other players’ characteristics \( \{a_k - c_k, k \neq i\} \). When \( c_i \) goes up, all the \( b_j, j = 1, 2, \cdots, n \) drop; therefore, all of the prices \( p_i^* \) rise. For player \( i \), the increment is stronger as the first term also becomes larger.

When two products are more substitutable (i.e., \( \beta \) increases), the firms are competing more intensively in the pricing game: the coefficient \( \frac{\beta}{2(2-\beta)} \) goes up as well as the decay factor \( \frac{2\delta}{2-\beta} \) in \( b_j \left( \mathbf{G}, \frac{2\delta}{2-\beta}, (\mathbf{a} - \mathbf{c}) \right) \). Therefore, in equilibrium, both firms give more generous discounts to
players, which leads to uniformly lower prices. Finally, the impact of a denser network is similar as $b_j \left( G, \frac{2\delta}{1-\beta}, (a-c) \right)$ is higher when $G$ is denser.

The above results allow us to understand the systematic change for all customers, but do not permit direct comparisons among the customers. To this end, we consider an asymptotic regime with sufficiently small network effects (i.e., when $\delta$ is sufficiently small). Using Taylor expansions, we can obtain more transparent expressions of the equilibrium prices.

**Theorem 2.** When $\delta$ is small, there is a unique Nash equilibrium whose prices are equal to

$$p^* = \frac{(1-\beta)a + c}{2-\beta} + \delta \left( \frac{-\beta}{(2-\beta)^2} G(a-c) \right) + O(\delta^2).$$

(11)

In other words, for each customer $i$,

$$p^A_i = p^B_i = \frac{(1-\beta)a_i + c_i}{2-\beta} - \frac{\delta\beta}{(2-\beta)^2} \sum g_{ij}(a_j - c_j) + O(\delta^2).$$

Furthermore, if we assume that $a_i = a, c_i = c$ for all $i$ (i.e. Assumption 3 holds), then

$$p^A_i = p^B_i = \frac{(1-\beta)a + c}{2-\beta} - \frac{\delta\beta(a-c)}{(2-\beta)^2} d_i + O(\delta^2),$$

(12)

where $d_i = \sum g_{ij}$ is the degree of customer $i$.

With Theorem 2 when $\delta$ is small, we obtain an intuitive pricing rule based on the degree-based pricing. Since the coefficient of $d_i$ is negative in (12), a customer with a higher degree will always obtain a lower price. In this respect, the firms compensate customers that are well-connected because their consumptions will boost other customers’ willingness to pay. Again this result has no counterpart in the monopoly case.

### 5.4 Examples

Let us now illustrate our main results using some specific network structures. We will first illustrate the results obtained in Theorem 1 where we showed that price competition between two firms leads to the fact that, in equilibrium, the structure of the network and the intensity of network effects matter in the price determination of the goods. We will then illustrate the comparative statics results of Proposition 2 especially the impact on the density of the network on equilibrium prices. In all of the examples in this section, we assume that Assumptions 1 and 2 hold, and thus we can apply Theorem 1 to compute the equilibrium prices. In some cases, we will impose a stronger condition by replacing Assumption 2 with Assumption 3.
5.4.1 The dyad: complete graph with 2 nodes (K₂)

Let us start with the simplest network, the dyad, which is the complete graph with only 2 consumers (denoted by K₂). It is displayed in Figure 1.

![Figure 1: The dyad](image)

As stated above, we adopt Assumption 2 so that, for \( i = 1, 2 \), \( a^A_i = a^B_i = a \) and \( c^A_i = c^B_i = c \). Using Theorem 1, we obtain the following equilibrium prices \( p^A = p^B = p^* \):

\[
p^* = \begin{bmatrix} p^*_1 \\ p^*_2 \end{bmatrix} = \left[ (2 - \beta)I_n - 2\delta G \right]^{-1} \left[ \left( (1 - \beta)I_n - \delta G \right)a + (I_n - \delta G)c \right] = \frac{1}{(2 - \beta)^2 - 4\delta^2} \begin{bmatrix} (1 - \beta)(2 - \beta) - 2\delta^2 & -\delta \beta \\ -\delta \beta & (1 - \beta)(2 - \beta) - 2\delta^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \frac{1}{2 - 2\delta - \beta} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
\]

In the dyad case, the network only plays a little role and thus firms do not discriminate consumers by their location in the network. It is easily verified that, when the marginal cost \( c_1 \) for serving customer 1 increases, equilibrium prices for both customers increase (\( \frac{\partial p^*_1}{\partial c_1} > 0 \) and \( \frac{\partial p^*_2}{\partial c_1} > 0 \)). By contrast, when \( a_1 \), the marginal intrinsic value of customer 1 increases, the equilibrium price for customer 1 increases, but the price for customer 2 decreases (\( \frac{\partial p^*_1}{\partial a_1} > 0 \) and \( \frac{\partial p^*_2}{\partial a_1} < 0 \)). Moreover, when \( a_1 = a_2 = a \), \( c_1 = c_2 = c \), we obtain:

\[
p^*_1 = p^*_2 = \frac{(1 - \delta - \beta)a + (1 - \delta)c}{2 - 2\delta - \beta}.
\]

We present some numerical examples in Table 1. As can be seen, even for this very simple network, the comparative statics results are not trivial. We start with the first row when the parameters are: \( a_1 = 3, a_2 = 4, c_1 = c_2 = 1, \beta = 0.4 \) and \( \delta = 0.2 \). Then, when we increase \( a_1 \) by 1 (second row), \( p^*_1 \) increases by 22.5 percent but \( p^*_2 \) decreases by only 1.6 percent. In other words, the effect of an increase of the marginal intrinsic value of customer 1 has different impact on prices.

Next, suppose that we increase \( c_2 \) by 1 (third row). In this case, both prices increase, but \( p^*_1 \) increases by only 2.1 percent while \( p^*_2 \) increase by 31.2 percent. When \( \beta \) increases or \( \delta \) increases, both prices decrease. These signs and percentage changes of prices are consistent with Proposition 2. All these results depend on the degree of substitution (or degree of product differentiation) \( \beta \) between the two goods and the network externalities \( \delta \).
### Regular graphs

We now consider the family of regular graphs. A network $G$ is regular of degree $d$ if each node has exactly $d$ neighbors, i.e., $G_{in} = d1_n$. Figure 2 displays an example of a regular graph of degree 2. For simplicity, we adopt here Assumption 3, i.e., $a_i = a_j = a, c_i = c_j = c$.

![Figure 2: A circle of four nodes $O_4$, which is also a regular graph of degree 2.](image)

Using Theorem 1, we obtain the following equilibrium prices for a regular graph of degree $d$:

$$
p^* = \left[ (2 - \beta)I_n - 2\delta G \right]^{-1} \left[ (1 - \beta)I_n - \delta G \right] a 1_n + (I_n - \delta G) c 1_n
\]
$$

$$
= c + \left( \frac{1 - d\delta - \beta}{2 - 2d\delta - \beta} \right) (a - c) 1_n.
$$

By differentiating this equation, we obtain $\frac{\partial p^*}{\partial a} > 0, \frac{\partial p^*}{\partial c} > 0$. Moreover,

$$
\frac{\partial p^*}{\partial \beta} = \frac{-d\delta(a - c)}{(2 - d\delta - \beta)^2}, \quad \frac{\partial p^*}{\partial \delta} = \frac{-d\beta(a - c)}{(2 - d\delta - \beta)^2}, \quad \frac{\partial p^*}{\partial d} = \frac{-\delta\beta(a - c)}{(2 - d\delta - \beta)^2}.
$$

As a result,

$$
\frac{\partial p^*}{\partial \beta} < 0, \text{sign} \left\{ \frac{\partial p^*}{\partial \delta} \right\} = -\text{sign} \{ \beta \}, \text{sign} \left\{ \frac{\partial p^*}{\partial d} \right\} = -\text{sign} \{ \beta \delta \}.
$$

When $\beta > 0$ and $\delta > 0$, the equilibrium price $p^*$ is decreasing in $\delta$ and $\beta$, and is also decreasing in the degree $d$. As above, these results are due to intensified competition between the two products.

---

### Table 1: Equilibrium prices for different parameters for the dyad network.

<table>
<thead>
<tr>
<th>$(a_1, a_2)$</th>
<th>$(c_1, c_2)$</th>
<th>$\beta$</th>
<th>$\delta$</th>
<th>$(p_1^<em>, p_2^</em>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 4)</td>
<td>(1, 1)</td>
<td>0.4</td>
<td>0.2</td>
<td>(1.633, 2.033)</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>(1, 1)</td>
<td>0.4</td>
<td>0.2</td>
<td>(2.0, 2.0)</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>(1, 2)</td>
<td>0.4</td>
<td>0.2</td>
<td>(1.667, 2.667)</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>(1, 1)</td>
<td><strong>0.5</strong></td>
<td>0.2</td>
<td>(1.498, 1.866)</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>(1, 1)</td>
<td>0.4</td>
<td><strong>0.3</strong></td>
<td>(1.545, 1.955)</td>
</tr>
</tbody>
</table>

The dyad network studied above is clearly a regular network of degree $d = 1$. 

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when each of these parameters increases. In particular, our last result says that the more connected consumers are (e.g. by having a denser network), the lower is the price paid for consuming the two goods.

5.4.3 The complete bipartite graph $K_{pq}$

Let us finally consider the complete bipartite graph, which is commonly used to model two-sided markets (see e.g. Ambrus and Argenziano [2009] and Jullien [2011]). In a complete bipartite graph $K_{mq}$, there are two disjoint groups $M$ and $Q$ such that any node in $M$ is connected to any node in $Q$. Let $m = |M|$ and $q = |Q|$. Then, the network size satisfies $n = m + q$. The adjacency matrix of a complete bipartite graph is given by: $G = \begin{bmatrix} 0 & J_{mq} \\ J_{qm} & 0 \end{bmatrix}$ where $J_{mq} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{m \times q}$. Figures 3 and 4 display two examples of bipartite networks for $m = 1, q = 5$ (Figure 3) and $m = 2, q = 3$ (Figure 4).

![Figure 3: A bipartite graph for $K_{15}$](image1)

![Figure 4: A bipartite graph for $K_{23}$](image2)

Using Theorem 1, the equilibrium prices can be expressed as (for a detailed proof, see Appendix B):

$$p^* = \frac{(1 - \beta)a + c}{2 - \beta} - \frac{\beta}{(2 - \beta)((2 - \beta)^2 - 4\delta^2 q m)} \left[ \frac{2\delta^2 q J_{mm}}{\delta(2 - \beta) J_{qm}} \frac{\delta(2 - \beta) J_{mq}}{2\delta^2 p J_{qq}} \right] (a - c).$$

(14)

Let us interpret equation (14). Consider two customers $i_1, i_2$ in one group of the bipartite graph. We find that $p^*_{i_1} > p^*_{i_2}$ if and only if $a_{i_1} > a_{i_2}$. Thus, within each group, the consumer with a higher intrinsic valuation of the good will be charged a higher price. However, there is no clear comparison for the prices across different groups. Moreover, under Assumption 3 (i.e. $a_i = a, c_i = c$), the equilibrium prices can be written as:

$$p^* = \frac{(1 - \beta)a + c}{2 - \beta} \mathbf{1}_n - \frac{\beta(a - c)}{(2 - \beta)((2 - \beta)^2 - 4\delta^2 q m)} \left[ \frac{(2\delta^2 pq + \delta(2 - \beta)q) \mathbf{1}_m}{(2\delta^2 pq + \delta(2 - \beta)p) \mathbf{1}_q} \right].$$
Therefore, in equilibrium, there are only two equilibrium prices, one for group \( M \) (denoted by \( p_M^* \)), the other for group \( Q \) (denoted by \( p_Q^* \)) where

\[
\begin{bmatrix}
p_M^*

p_Q^*
\end{bmatrix} = \frac{1}{2 - \beta} \left[ \frac{(1 - \beta)a + c - \frac{\beta(a-c)(2\delta^2mq + \delta(2-\beta)q)}{(2-\beta)^2 - 4\delta^2qm}}{(1 - \beta)a + c - \frac{\beta(a-c)(2\delta^2mq + \delta(2-\beta)m)}{(2-\beta)^2 - 4\delta^2qm}} \right].
\] (15)

As a result, the price difference between group \( P \) and \( Q \) is equal to:

\[
p_M^* - p_Q^* = \frac{\beta\delta(2-\beta)(a-c)}{(2-\beta)((2-\beta)^2 - 4\delta^2qm)} \times (m - q).
\]

Therefore, we obtain that

\[
p_M^* > p_Q^* \text{ if and only if } |M| = m > |Q| = q.
\]

Applying this result to the star network (where \( m = 1 \)), we conclude that the consumer located in the center is charged with a lower price than the consumers located at the periphery.

**The impact of network structure on equilibrium prices.** Consider the following two networks. It can be seen that adding one link between nodes 2 and 3 in the star network \( K_{12} \) in Figure 5 leads to the complete network \( K_3 \) in Figure 6.

![Figure 5: star \( K_{12} \)](image)

![Figure 6: \( K_3 \)](image)

By Proposition 2, the equilibrium price for every customer in the complete network \( K_3 \) is lower than in the star network \( K_{12} \) for any parameter value. This is because the network \( K_3 \) is denser than the network \( K_{12} \). This can be seen by comparing the second and third columns in Table 2 where we have calculated the equilibrium prices for these two networks for specific parameter values. Furthermore, for the complete network \( K_3 \), the network position of every node is the same and, thus, a higher marginal utility \( a_i \) means a higher price \( p_i^* \) (see third row in Table 2). For the star network \( K_{12} \), when all consumers have the same \( a \), the customer with higher (Katz-Bonacich) centrality will have a lower price. Indeed, the price for the center customer 1 in \( K_{12} \) is \( p_1^* = 1.764 \), which is lower than \( p_2^* = p_3^* = 1.790 \). In the last row of Table 2, customer 1 has the largest marginal utility of consuming the product. When the network is the star \( K_{12} \), her price is the
lowest. However, when customers 2 and 3 form a link, i.e., when the network becomes complete, it will then be customer 1 who will experience the highest price. This is because, in the star network, the central position of individual 1 "compensates" for her strong willingness to pay for the product. This is clearly not anymore the case in the complete network where 1 has no positional advantage and thus does not generate more network externalities than the other consumers. This highlights the key trade off that firms face when deciding on their prices. They have some monopoly power over consumers who have a strong preference for consuming the good but they also need to take into account how much network externalities each consumer generates, which is captured by the individual’s network centrality.

\[
(a_1, a_2, a_3) \quad \begin{array}{ccc}
K_{12} & p^*_1 & p^*_2 & p^*_3 \\
(3, 3, 3) & 1.764 & 1.790 & 1.790 \\
(4, 3, 3) & 2.172 & 1.777 & 1.777 \\
(3, 4, 3) & 1.751 & 2.200 & 1.788 \\
(3.03, 3.01, 3) & 1.776 & 1.794 & 1.790 \\
\end{array}
\]

\[
(a_1, a_2, a_3) \quad \begin{array}{ccc}
K_3 & p^*_1 & p^*_2 & p^*_3 \\
(3, 3, 3) & 1.754 & 1.754 & 1.754 \\
(4, 3, 3) & 2.162 & 1.739 & 1.739 \\
(3, 4, 3) & 1.739 & 2.162 & 1.739 \\
(3.03, 3.01, 3) & 1.766 & 1.758 & 1.753 \\
\end{array}
\]

Table 2: Equilibrium prices for two different networks. The parameters are \(c_i = 1, i = 1, 2, 3\), \(\beta = 0.3\) and \(\delta = 0.12\).

6 Equilibrium profits

In this section, we determine the firms’ profit and how it varies with some parameters. For the ease of exposition, we will impose Assumption 2 in this section.

6.1 Equilibrium profits

Let us define

\[
\phi(z) := \frac{(1 - \delta z)(1 - \beta - \delta z)}{(1 + \beta - \delta z)(2 - \beta - 2\delta z)^2}.
\] (16)

Then the profit function can be rewritten as

\[
\Pi = \langle (a - c), \phi(G)(a - c) \rangle.
\]

We have the following result:

Proposition 3. In equilibrium, each firm obtains a profit equal to

\[
\Pi^* = \langle (a - c), [(1 + \beta)I_n - \delta G]^{-1}[(2 - \beta)I_n - 2\delta G]^{-2}[I_n - \delta G][(1 - \beta)I_n - \delta G](a - c) \rangle.
\] (17)
We can now derive some comparative statics results of the firms’ equilibrium profits. For a fixed graph $G$, let $D$ be the pair $(\beta, \delta)$ that satisfies Assumption 1, i.e.,

$$D = \{(\beta, \delta) | \delta \geq 0, 1 - |\beta| - \delta \lambda_1(G) > 0\}.$$  

**Proposition 4.** For any network $G$ and for any $(\beta, \delta)$ in the domain $D$, the equilibrium profit is positive, i.e., $\Pi^* > 0$. Moreover, the equilibrium profit is always decreasing in $\beta$, i.e., $\frac{\partial \Pi^*}{\partial \beta} < 0$.

Note, however, that the sign of $\frac{\partial \Pi^*}{\partial \delta}$ could be positive or negative, depending on the parameters of the model and the network, i.e. $G, \delta, \beta$. In the proof of Proposition 4, we provide concrete examples illustrating this indeterminacy. It is worth mentioning that, in deriving Proposition 4, we do not impose any assumption on $G, a, c$. Thus, the results are quite robust.

### 6.2 Regular graphs

We now consider regular graphs of degree $d$. For simplicity, we adopt Assumption 3 (FS), i.e., $a_i = a, c_i = c, \forall i$. The equilibrium price is then given by:

$$p^* = \frac{(1 - d\delta - \beta)a + (1 - d\delta)c}{2 - 2d\delta - \beta},$$

for all customers, and the equilibrium profit function is:

$$\Pi^* = n(a - c)^2 \frac{(1 - d\delta)(1 - \beta - d\delta)}{(2 - \beta - 2d\delta)^2(1 + \beta - \delta d)}.$$

where $\frac{\partial \Pi^*}{\partial a} > 0$, and $\frac{\partial \Pi^*}{\partial c} < 0$. Furthermore, we can show that

$$\frac{\partial \Pi^*}{\partial \beta} = n(a - c)^2 \frac{2(1 - d\delta)((1 - d\delta)^2 - (1 - d\delta)\beta + \beta^2)}{(2 - \beta - 2d\delta)^3(1 + \beta - \delta d)^2} < 0$$

as $(1 - d\delta)^2 - (1 - d\delta)\beta + \beta^2 = (1 - d\delta - \frac{\beta}{2})^2 + \frac{3}{4}\beta^2 > 0$. On the other hand,

$$\frac{\partial \Pi^*}{\partial \delta} = n(a - c)^2 \times \frac{-d(\beta^3 + 3(1 - d\delta)^2\beta - 2(1 - d\delta)^3)}{(2 - \beta - 2d\delta)^3(1 + \beta - \delta d)^2}.$$

Note that $\beta^3 + 3(1 - d\delta)^2\beta - 2(1 - d\delta)^3 > 0$ if and only if $\beta > \beta^*(1 - d\delta)$, where $\beta^* \approx 0.596072$ is the unique real root of $2 - 3\beta - \beta^3$. Therefore,

$$\frac{\partial \Pi^*}{\partial \delta} > 0 \text{ if and only if } \beta < \beta^*(1 - d\delta).$$

Therefore, the sign of $\frac{\partial \Pi^*}{\partial \delta}$ will depend on the parameters of the model.

Let us better understand the impact of $\delta$ on the equilibrium profit. First, for a fixed price
vector, increasing $\delta$ leads to a higher intensity of network effects, and, as a result, the customers consume more since they are more influenced by their neighbors. Second, increasing $\delta$ leads to a more intensified price competition. This price effect drives down the equilibrium prices and can potentially reduce the firms’ profits. For fixed $\delta$, when products are highly differentiated ($\beta$ small), the network effect dominates the price effect. Therefore, the equilibrium profit is increasing in $\delta$. However, when $\beta$ is relative large, the price effect dominates and the equilibrium profit actually decreases with $\delta$. Recall that for a monopoly firm, the price effect disappears because $p = \frac{a+c}{2}$, which does not vary with $\delta$. Moreover, $\beta = 0$ is equivalent to the monopoly case. When $\beta = 0$, the condition $\beta < \beta^*(1 - d\delta)$ is trivially satisfied for any $\delta$. Therefore, increasing $\delta$ is always beneficial for a monopoly firm, but it could be detrimental in the presence of product competition.

Next, we investigate the impact of the degree $d$ on the equilibrium profit. Intuitively, this should be the same as increasing $\delta$ since:

$$\frac{\partial \Pi^d}{\partial d} = n(a - c)^2 \frac{-\delta(\beta^3 + 3(1 - d\delta)^2\beta - 2(1 - d\delta)^3)}{(2 - \beta - 2d\delta)^3(1 + \beta - d\delta)^2}.$$  

We observe that, when $\delta > 0$,

$$\frac{\partial \Pi^d}{\partial d} > 0 \text{ if and only if } \frac{\partial \Pi^d}{\partial \delta} > 0 \text{ if and only if } \beta < \beta^*(1 - d\delta).$$

Therefore, the sign of $\partial \Pi^d/\partial d$ could be positive or negative, depending on the parameters. Moreover, a denser network could be detrimental for the competing firms. Again this is in contrast with the case of monopoly firms where $\beta = 0$. We can summarize our findings in the following proposition.

**Proposition 5.** For regular graphs of degree $d$:

1. When $\beta = 0$ (the monopoly case), increasing $\delta$ or increasing $d$ always benefits the firms in terms of profits.

2. When $\beta > 0$ (duopoly case), in certain parameter range, increasing $\delta$ or increasing $d$ can hurt competing firms.

Let us provide some graphic illustrations of these results. For $d = 2$, in Figure 7, we plot the range for which $\Pi^*$ is increasing or decreasing in $\delta$. Notably, in most of the analysis, we assume $\beta > 0, \delta > 0$. However, almost all of the results about equilibrium prices and profits are valid for negative $\beta$ or negative $\delta$, as long as the stability condition (Assumption 1) is satisfied. Therefore, we plot both figures in the examples, one for the positive quadrant, the other for the whole parameter range. In Figure 7, the horizontal axis corresponds to the increase in $\beta$ while the vertical axis corresponds to the increase in $\delta$. Moreover, the red region is when $\frac{\partial \Pi^*}{\partial \delta} > 0$ while the blue region is when $\frac{\partial \Pi^*}{\partial \delta} < 0$. In the left panel, we see that, even when $\delta$ and $\beta$ are negative, an increase in $\delta$
always increase the equilibrium profit. On the right panel, where only positive values of \( \delta \) and \( \beta \) are considered, we see that an increase in \( \delta \) can lead to a decrease in equilibrium profits.

We can also investigate the impact of network density on equilibrium profit. For that, we compare the circle network \( O_4 \) and the complete network \( K_4 \) with four agents. Both are regular graphs with 4 nodes, but they differ in their degrees since, for the circle network, the degree is two while, for the complete graph, it is three. Let us compare the equilibrium price and profit for these two graphs. In Figure 8 the horizontal axis represents \( \beta \) while the vertical axis corresponds to \( \delta \). Moreover, the red region is when firms charge a higher price in the complete network \( K_4 \) while, in the blue region, firms charge a higher price in the circle network \( O_4 \). In this figure, both \( \delta \) and \( \beta \) can be positive or negative.

We can see that, when \( \beta > 0, \delta > 0 \) (our main focus), the price effect drives the price down. Indeed, since the complete network \( K_4 \) is denser than the circle network \( O_4 \), then the price under \( O_4 \) is higher. On the other hand, for the profit comparison (see Figure 9), when \( \beta > 0 \) and \( \delta > 0 \), the profit of firms is higher in \( K_4 \) compared to \( O_4 \) only when the parameter \( \delta \) is below a certain cut-off curve of \( \beta \). In particular, for very small \( \beta \), profits in the complete network \( K_4 \) are always higher than in the circle network \( O_4 \) for almost all \( \delta > 0 \) in the feasible domain.

### 6.3 General graphs

In fact, the findings of the previous section can also be obtained even for general graphs. Recall that \( G' \succ G \) if \( G' \) contains \( G \) as a subgraph. Let

\[
D' = \{(\beta, \delta) | 0 \leq \beta < 1, \delta \geq 0, 1 - |\beta| - |\delta| \lambda_1(G') > 0\}
\]

be the feasible domain for the parameter \( (\beta, \delta) \) under \( G' \).
Figure 8: Comparing prices between the star and the complete network with four consumers.

**Proposition 6.** Suppose $G' \succ G$. Then,

1. There exists a nonempty open subset $\Theta_1$ of $D'$ such that for any parameters $(\beta, \delta)$ in this open set, the equilibrium profit under $G'$ is higher than that under $G$:

   $$\Pi^*(G') > \Pi^*(G), \quad \forall (\beta, \delta) \in \Theta_1.$$ 

2. There exists a nonempty open subset $\Theta_2$ of $D'$ such that for any parameters $(\beta, \delta)$ in this open set, the equilibrium profit under $G'$ is lower than that under $G$:

   $$\Pi^*(G') < \Pi^*(G), \quad \forall (\beta, \delta) \in \Theta_2.$$ 

The key message of Proposition 6 is that the equilibrium profit $\Pi^*$ is very sensitive to the parameters of the model, in particular, $\beta$, the degree of substitution between the two products $A$ and $B$. This suggests that many intuitive marketing strategies such as enhancing the network effects (increasing $\delta$) or making customers more connected to each other may backfire and hurt the firms that compete in the market. Blindly applying the results from the monopoly pricing setup can be risky because the implications do not necessarily carry over to the competitive pricing in networks.

---

$^9$Similarly, one can show that, for a fixed graph $G$, there exists a nonempty subset $\Theta_1$ of $D$ such that $\Pi^*$ is increasing in $\delta$ in this open set $\Theta_1$, while there exists another nonempty subset $\Theta_2$ of $D$ such that $\Pi^*$ is decreasing in $\delta$ in this open set $\Theta_2$. 

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7 Extensions

This section extends the model in several directions.

7.1 Oligopolistic Competition: more than two firms

In this section, we extend our analysis to accommodate more than two firms. Suppose that \( l \geq 2 \) firms engage in price competition. There are \( n \) consumers and \( l \) firms and each consumer consumes the \( l \) goods and each firm \( t = 1, \ldots, l \) produces one good. Since each individual now consumes \( l \) goods, the utility function can now be written as:

\[
    u_i(x) = \sum_{t=1}^{l} a_t^i x_t^i - \frac{1}{2} \sum_{t=1}^{l} (x_t^i)^2 - \frac{1}{2} \sum_{t=1}^{l} \sum_{s \neq t} \beta x_t^i x_s^i \tag{18}
\]

\[
    + \delta \sum_{t=1}^{l} \left( \sum_{j=1}^{n} g_{ij} x_j^t x_j^t \right) - \sum_{t=1}^{l} p_t^i x_t^i.
\]

We can define the interdependence matrix as:

\[
    \Psi = \begin{bmatrix}
    1 & \beta & \cdots & \beta \\
    \beta & 1 & \cdots & \beta \\
    \vdots & \ddots & \ddots & \vdots \\
    \beta & \cdots & \beta & 1
    \end{bmatrix}.
\]

As above, the parameter \( \beta \) lies in the interval \((0, 1)\). From (18), we can see that network externalities operate only among the same products.
For simplicity, we make the following semi-symmetry (SS) assumption.

**Assumption 4.** \( a^t = a^s = a \) and \( c^t = c^s = c \) for any two firms \( s \) and \( t \).

Define
\[
\Phi(z) := \frac{(1 + (l - 2) \beta - \delta z)(1 - \beta - \delta z)}{(1 + (l - 1) \beta - \delta z)(2 + (l - 3) \beta - 2 \delta z)^2}.
\]

**Theorem 3.** Consider an oligopoly model with \( l \geq 2 \) firms and suppose that Assumptions 1 and 4 hold. Then, there exists a unique equilibrium in the pricing stage in which all firms charge the same price \( p^* \) defined as follows:
\[
p^* = a + c - \frac{(l - 1) \beta}{2}[2 + (l - 3) \beta]^{-1}(a - c) \tag{20}
\]
In this equilibrium, for each firm, the consumption vector is equal to:
\[
x^* = [(1 + (l - 1) \beta)I_n - \delta G]^{-1}[2 + (l - 3) \beta]I_n - 2 \delta G]^{-1}(a - c),
\]
and the equilibrium profit is given by:
\[
\Pi = \langle x^*, (p^* - c) \rangle = \langle (a - c), \Phi(G)(a - c) \rangle. \tag{21}
\]

We have a similar decomposition as in the duopoly case. Moreover, the comparative statics results on \( a, c, \delta, \beta \) and the network structure are also similar, and we hereby omit the details.\(^{10}\)

To avoid repetition, we focus on the impact of \( l \), the number of firms, on outcomes.

First, observe that the formula in (20) works even for \( l = 1 \). In the second term
\[
\Sigma \equiv \frac{(l - 1) \beta}{2[2 + (l - 3) \beta]} b \left( G, \frac{2 \delta}{2 + (l - 3) \beta}, (a - c) \right),
\]
the coefficient is zero only if \( l = 1 \) (monopoly) or \( \beta = 0 \) (independent products). Moreover, when \( \beta > 0 \), the decay factor \( \frac{2 \delta}{2 + (l - 3) \beta} \) is decreasing in the number of firms \( l \), and goes to zero as \( l \) goes to infinity. In the limit, we have
\[
\lim_{l \to +\infty} \Sigma = \begin{cases} 
\frac{a - c}{2} & \beta > 0, \\
0 & \beta = 0.
\end{cases}
\]
As a consequence, we obtain the following limiting results.

**Proposition 7.** In the limit, for any network \( G \),
\[
\lim_{l \to +\infty} p^* = \begin{cases} 
c, & \beta > 0; \\
\frac{a + c}{2}, & \beta = 0.
\end{cases}
\]

\(^{10}\)The results and proofs are available upon request. Also, in Appendix D we provide additional results for the oligopoly case, especially for regular graphs.
The corresponding profits are:

\[
\lim_{t \to +\infty} \Pi = \begin{cases} 
0, & \beta > 0; \\
\frac{1}{4}((a - c), [I_n - \delta G]^{-1}(a - c)), & \beta = 0.
\end{cases}
\]

For competitive markets with infinitely many firms, the structure of the network has little impact on equilibrium prices. The only thing that matters is \( \beta \), the degree of substitution between products. This is because the intense competition drives away firms’ pricing power. In particular, in this proposition, we show that, when \( \beta = 0 \), each firm earns a positive profit in equilibrium but, when \( \beta > 0 \), as the number of firms increases, the equilibrium profit goes to zero.

### 7.2 Uniform prices

Let us go back to the duopoly case with two products \( A \) and \( B \). In the previous analysis, we have assumed that both firms can fully discriminate over customers in the network by charging individual prices. In this extension, we require them to charge uniform prices to all customers. To see how this restriction affects the analysis, note that in a symmetric equilibrium with uniform prices, both firms choose \( p^* A = p^* B = p^u 1_n \). Thus, the induced consumptions are \( x^A = x^B = x^* = M^+(a - p^u 1_n) \) by Corollary 1 and, accordingly, the profits are \( \Pi^A = \Pi^B = \langle x^*, p^u 1_n - c \rangle \). We obtain the following result.

**Theorem 4.** Suppose both firms are restricted to charge uniform prices for all customers. There exists a unique equilibrium in the pricing stage where both firms charge \( p^u 1_n \), where

\[
p^u = \frac{\langle 1_n, (M^+ a + \frac{M^+ + M^-}{2} c) \rangle}{\langle 1_n, (M^+ 1_n + \frac{M^+ + M^-}{2} 1_n) \rangle} = \frac{\langle 1_n, [(1 + \beta)I_n - \delta G]^{-1}(2a + c) + \langle 1_n, [(1 - \beta)I_n - \delta G]^{-1}c \rangle}{\langle 1_n, 3[(1 + \beta)I_n - \delta G]^{-1}1_n \rangle}.
\]

The derivation of this theorem is similar to that of Theorem 1. We observe that unless certain strong assumptions are satisfied, the symmetric discriminatory price vectors derived in Theorem 1 and the uniform price vector derived in Theorem 4 are different. Therefore, the competitive firms do exercise their price discrimination power at very refined levels. More importantly, even if the firms are restricted to charge uniform prices, the matrix decomposition (via \( M^+ \) and \( M^- \)) still plays a crucial role in the market equilibrium.

### 7.3 Cross network effects

It is sometimes natural to allow for cross network effects, i.e., the network externality can exist between two different products. This means that the consumption of product \( A \) by individual \( i \)
has a direct positive impact of the consumption of product $B$ by individual $j$ if $i$ and $j$ are directly connected in the network. The utility function is now given by where we allow that the intensity of the network effects to be different between and within products:

\[
  u_i(x_i, x_{-i}) = a_i x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2} (x_i^A)^2 + \frac{1}{2} (x_i^B)^2 + \beta x_i^A x_i^B \right\} \\
  + \delta \sum_{j=1}^n g_{ij} x_i^A x_j^A + \delta \sum_{j=1}^n g_{ij} x_i^B x_j^B + \mu \sum_{j=1}^n g_{ij} x_i^A x_j^B + \mu \sum_{j=1}^n g_{ij} x_i^B x_j^A \\
  - p_i^A x_i^A - p_i^B x_i^B.
\]  

Note that we assume that the network is the same so that people consuming different goods belong to the same network. However, we add the cross externality terms:

\[
  \delta \sum_{j=1}^n g_{ij} x_i^A x_j^A + \delta \sum_{j=1}^n g_{ij} x_i^B x_j^B + \mu \sum_{j=1}^n g_{ij} x_i^A x_j^B + \mu \sum_{j=1}^n g_{ij} x_i^B x_j^A,
\]

with $\mu \geq 0$ capturing the intensity of cross network effects. We also assume that the cross network effect is weaker than direct network effect, i.e., $\mu \leq \delta$. We can now characterize the symmetric equilibrium as follows.

**Theorem 5.** Suppose that Assumption 2 and the following stability condition hold:

\[
(1 + \beta) - |\delta + \mu| \lambda_1(G) > 0, \quad \text{and} \quad (1 - \beta) - |\delta - \mu| \lambda_1(G) > 0.
\]

Then, there exists a unique equilibrium where both firms choose the following prices:

\[
p^{cr} = \left[ (2 \beta) I_n - (2 \delta - \mu) G \right]^{-1} \left[ ((1 - \beta) I_n - (\delta - \mu) G) a + (I_n - \delta G) c \right].
\]  

Let us illustrate how we can apply Theorem 5. For regular network with degree $d$, suppose that Assumption 3 holds. We obtain that

\[
p^{cr} = p^{cr} 1_n,
\]

where

\[
p^{cr} = \frac{((1 - \beta) - (\delta - \mu)d) a + (1 - \delta d) c}{((2 - \beta) - (2 \delta - \mu)d)} = c + \frac{((1 - \beta) - (\delta - \mu)d)}{((2 - \beta) - (2 \delta - \mu)d)} (a - c).
\]

It is then easily verified that

\[
\frac{\partial p^{cr}}{\partial \mu} = \frac{(1 - \delta) d}{((2 - \beta) - (2 \delta - \mu)d)^2} (a - c) > 0.
\]

This suggests that cross network effects soften the competition between firms, and because of this, the equilibrium price is higher.

Next, we show that the result does not depend on the regular network and strong (FS) assumption (Assumption 3). Formally, we can show the following.

**Proposition 8.** When cross-network effect intensity $\mu$ increases, the competitive prices always go
Indeed, \[ \frac{\partial p^{cr}}{\partial \mu} = [(2 - \beta)I_n - (2\delta - \mu)G]^{-2}(1 - \delta)G(a - c) \geq 0. \]

As a consequence, when \( \mu = 0 \) (Theorem 1), equilibrium prices are lower than when \( \mu \) is strictly positive.

In Table 3, we list the equilibrium prices for the star network with three consumers (network \( K_{12} \)) for different values of \( \mu \), and, indeed, for every customer, we can see that the equilibrium price is increasing in \( \mu \). In contrast, one can check that the collusive price is the same as before for any \( \delta \) and \( \mu \) because it does not depend on the network structure.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( p_1^{cr} )</th>
<th>( p_2^{cr} )</th>
<th>( p_3^{cr} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>2.185</td>
<td>1.780</td>
<td>1.367</td>
</tr>
<tr>
<td>0.05</td>
<td>2.254</td>
<td>1.840</td>
<td>1.428</td>
</tr>
<tr>
<td>0.10</td>
<td>2.314</td>
<td>1.896</td>
<td>1.485</td>
</tr>
<tr>
<td>0.11</td>
<td>2.325</td>
<td>1.907</td>
<td>1.495</td>
</tr>
</tbody>
</table>

Table 3: Equilibrium prices for the star network \( K_{12} \) with different \( \mu \)s for \( a = (4, 3, 2) \), \( c = (1, 1, 1) \), \( \beta = 0.3 \), and \( \delta = 0.12 \).

### 7.4 Asymmetric firms

So far, we have assumed that each customer has the same intrinsic marginal utility for different products. This assumption implies that the equilibrium prices for different products are also the same for a fixed customer. In this subsection, we remove this symmetry assumption and solve for equilibrium prices with heterogeneous intrinsic marginal utilities and marginal costs. For simplicity, we concentrate on the duopoly case. The analysis for oligopoly is similar, and is left for interested readers. We have the following result:

**Theorem 6.** Suppose that Assumption 1 holds. Then, for any \( a^A, a^B \) and \( c^A, c^B \), there exists a unique equilibrium in prices \( (\hat{p}^A, \hat{p}^B) \) that satisfies:

\[
\begin{align*}
\hat{p}^A &= \frac{a^A + c^A}{2} - \frac{\beta}{2(2-\beta)}b \left(G, \frac{2\delta}{2-\beta}, \left(\frac{a^A + a^B}{2} - \frac{c^A + c^B}{2}\right)\right) + \frac{\beta}{2(2+\beta)}b \left(G, \frac{2\delta}{2+\beta}, \left(\frac{a^A - a^B}{2} - \frac{c^A - c^B}{2}\right)\right), \\
\hat{p}^B &= \frac{a^B + c^B}{2} - \frac{\beta}{2(2-\beta)}b \left(G, \frac{2\delta}{2-\beta}, \left(\frac{a^A + a^B}{2} - \frac{c^A + c^B}{2}\right)\right) - \frac{\beta}{2(2+\beta)}b \left(G, \frac{2\delta}{2+\beta}, \left(\frac{a^A - a^B}{2} - \frac{c^A - c^B}{2}\right)\right).
\end{align*}
\]

We see that the equilibrium prices depend of the average marginal willingness to pay for the products and on the position of the customers in the network as captured by their Katz-Bonacich centralities.
8 Conclusion

In this paper, we consider a duopoly setting in which two firms sell products to the customers in a social network. The two products are interdependent: they can be either substitutable or complementary. Customers are endowed with heterogeneous intrinsic valuations for the products. Moreover, there are local network externalities amongst the customers in terms of their consumption utilities. We provide a full characterization of the equilibrium prices for any network structure, and relate these equilibrium outcomes to the familiar Katz-Bonacich centrality measures. In contrast to the monopoly case, we show that the equilibrium prices exhibit strong network dependence, which implies that the knowledge of the network structure is crucial for profit maximization. We provide some examples of networks that illustrate how the competitive prices depend upon the customers’ relative positions. We also show that when the firms sell substitutable products, enhancing network externalities among customers actually pushes the equilibrium price downwards. Moreover, we show that firms’ equilibrium profits can be reduced when either the network becomes denser or the network effect is strengthened. Therefore, competition can lead to substantially different implications of the pricing strategies as well as firms’ profitability. Finally, we extend our analysis to accommodate oligopoly competition of more than two firms, and show that the results can be generalized.

References


Appendix

In this Appendix, we first provide some notations, derive some matrix operations, then give the proofs of our main results. We also analyze the single representative customer benchmark (benchmark 1 in Section 3). Finally, we offer some additional results in the oligopoly model (Section 7.1).

A Matrix notations and Katz-Bonacich centrality

A.1 Matrix notations

Let us have some notations for the matrices and vectors in general. In this paper, $I_k$ is the $k \times k$ identity matrix, $J_{pq}$ is the $p \times q$ matrix with 1’s, and $1_n = J_{n1}$ is a column vector with 1s:

$$I_k = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{k \times k}, \quad J_{pq} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{p \times q}, \quad 1_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}.$$

The inner product of two vectors $x = (x_1, \ldots, x_n)'$ and $y = (y_1, \ldots, y_n)'$ in $\mathbb{R}^n$ is denoted as $\langle x, y \rangle = \sum_i x_i y_i$. We use $0$ to denote the zero matrix with suitable dimensions. For any two matrices $H$ and $D$, $H \preceq (\succeq) D$ if component-wise $h_{ij} \leq (\geq) d_{ij}$ for all $i,j$, where $\{h_{11}, \ldots, h_{mn}\}$'s and $\{d_{11}, \ldots, d_{mn}\}$'s are the components of $H$ and $D$, respectively. Consequently, $H$ is a positive matrix if $H \succeq 0$. $H'$ represents the transpose of a matrix $H$. A square symmetric matrix $H$ is called positive definite if all of its eigenvalues are strictly positive. For two matrices $H_1$ and $H_2$, their Kronecker product is:

$$H_1 \otimes H_2 = \begin{bmatrix} h_{11}H_2 & \cdots & h_{1t}H_2 \\ \vdots & \ddots & \vdots \\ h_{s1}H_2 & \cdots & h_{st}H_2 \end{bmatrix}.$$

We define $\otimes$ as a bi-linear operator such that for any matrices $H_1, H_2, H_3,$ and $H_4$,

$$(H_1 \otimes H_2)(H_3 \otimes H_4) = (H_1H_3) \otimes (H_2H_4), \quad (H_1 \otimes H_2)^{-1} = H_1^{-1} \otimes H_2^{-1}, \quad (H_1 \otimes H_2)' = (H_1' \otimes H_2').$$

A.2 Katz-Bonacich centrality

Let us define the Katz-Bonacich centrality. Denote by $\lambda_1(G)$ the spectral radius of matrix $G$. Since $G$ is a nonnegative matrix, by the Perron-Frobenius Theorem it is also equal to its largest eigenvalue.
**Definition 1.** Assume $0 \leq \delta < 1/\lambda_1(G)$. Then, for any vector $a = (a_1, \cdots, a_n)' \in \mathbb{R}^n$, the Katz-Bonacich centrality vector with weight $a$ is defined as:

$$b(G, \delta, a) := M(G, \delta)a,$$

where

$$M(G, \delta) = [I - \delta G]^{-1} = I + \sum_{k \geq 1} \delta^k G^k.$$

Let $b_i(G, \delta, a)$ be the $i$th entry of $b(G, \delta, a)$. Let $m_{ij}(G, \delta)$ be the $ij$ entry of $M(G, \delta)$. Then,

$$b_i(G, \delta, a) = \sum_j m_{ij}(G, \delta)a_j.$$

**B Proofs**

In this section, we provide the detailed proofs of our technical results.

**Proof of Proposition 1** Omitted.

**Proof of Corollary 1** When $a^A = a^B = a$ and $p^A = p^B = p$, by (4) we have:

$$x^A = M^+ \left( \frac{a^A + a^B}{2} - \frac{p^A + p^B}{2} \right) + M^- \left( \frac{a^A - a^B}{2} - \frac{p^A - p^B}{2} \right) = 0 \text{ as } a^A = a^B, \text{ and } p^A = p^B.$$

By the same logic, $x^B = M^+(a - p)$.

**Proof of Lemma 1** We can express the joint profit as

$$\pi^A(p^A, p^B) + \pi^B(p^A, p^B) = \begin{bmatrix} p^A - c^A & p^B - c^B \end{bmatrix} \begin{bmatrix} x^A(p^A, p^B) \\ x^B(p^A, p^B) \end{bmatrix} = \begin{bmatrix} p^A - c^A & p^B - c^B \end{bmatrix} \begin{bmatrix} M^+ + M^- \\ M^+ - M^- \end{bmatrix} \begin{bmatrix} a^A - p^A \\ a^B - p^B \end{bmatrix}.$$

The corresponding first-order conditions are

$$\left[ \begin{array}{cc} M^+ + M^- & M^+ - M^- \\ M^+ - M^- & M^+ + M^- \end{array} \right] \begin{bmatrix} a^A - \tilde{p}^A \\ a^B - \tilde{p}^B \end{bmatrix} - \left[ \begin{array}{cc} M^+ + M^- & M^+ - M^- \\ M^+ - M^- & M^+ + M^- \end{array} \right] \begin{bmatrix} \tilde{p}^A - c^A \\ \tilde{p}^B - c^B \end{bmatrix} = 0. \quad (27)$$
Notice that matrix
\[
\begin{bmatrix}
\frac{M^+ + M^-}{2} & \frac{M^+ - M^-}{2} \\
\frac{M^+ - M^-}{2} & \frac{M^+ + M^-}{2}
\end{bmatrix}
\]
is symmetric. Therefore, we obtain
\[
\begin{bmatrix}
\frac{M^+ + M^-}{2} & \frac{M^+ - M^-}{2} \\
\frac{M^+ - M^-}{2} & \frac{M^+ + M^-}{2}
\end{bmatrix}' = \begin{bmatrix}
\frac{M^+ + M^-}{2} & \frac{M^+ - M^-}{2} \\
\frac{M^+ - M^-}{2} & \frac{M^+ + M^-}{2}
\end{bmatrix}.
\]
As a result, (27) can be simplified to
\[
\begin{bmatrix}
\frac{M^+ + M^-}{2} & \frac{M^+ - M^-}{2} \\
\frac{M^+ - M^-}{2} & \frac{M^+ + M^-}{2}
\end{bmatrix}
\begin{bmatrix}
aA + cA - 2\bar{p}A \\
aB + cB - 2\bar{p}B
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
Recall that the eigenvalues of
\[
\begin{bmatrix}
\frac{M^+ + M^-}{2} & \frac{M^+ - M^-}{2} \\
\frac{M^+ - M^-}{2} & \frac{M^+ + M^-}{2}
\end{bmatrix}
\]are \(\lambda_i(G)\), \(i = 1, \ldots, n\), which are positive by Assumption 1. Hence, it is an invertible matrix. This then leads to:
\[
\begin{bmatrix}
aA + cA - 2\bar{p}A \\
aB + cB - 2\bar{p}B
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix}\bar{p}A \\ \bar{p}B\end{bmatrix} = \begin{bmatrix} \frac{aA + cA}{2} \\ \frac{aB + cB}{2} \end{bmatrix}.
\]
The result just follows.

Proof of Theorem 1: The symmetric equilibrium is constructed in the main text. From (10), we obtain that
\[
p^* = (3M^+ + M^-)^{-1} \left\{ 2M^+a + (M^+ + M^-)c \right\}
\]
\[
= [(2 - \beta)I_n - 2\delta G]^{-1} \left\{ ((1 - \beta)I_n - \delta G)a + (I_n - \delta G)c \right\}
\]
\[
= c + [(2 - \beta)I_n - 2\delta G]^{-1} [(1 - \beta)I_n - \delta G](a - c)
\]
\[
= a + c - \frac{\beta}{2} (2 - \beta)I_n - 2\delta G]^{-1}(a - c).
\]
Moreover, it is the unique equilibrium because Theorem 1 is just a special case of Theorem 6. The complete proof is relegated to that of Theorem 6.

Proof of Proposition 2: Recall that, by (9)
\[
p^* = \frac{a + c}{2} - \frac{\beta}{2} (2 - \beta)I_n - 2\delta G]^{-1}(a - c).
\]
When \(\delta = 0\) (no network effect), \(p^* = \frac{a + c}{2} - \frac{\beta}{2(2 - \beta)}(a - c) = \frac{(1 - \beta)a + c}{2 - \beta}\). When \(\beta = 0\), (two independent networks), \(p^* = \frac{a + c}{2}\), which clearly is independent of parameter \(\delta\) and network \(G\).

For the second part, first we observe that
\[
\frac{\partial p^*}{\partial \delta} \bigg|_{\delta=0} = -\frac{\beta}{(2 - \beta)^2} G(a - c).
\]
If $p^*$ is independent of $\delta$, $-\frac{\beta}{2(2 - \beta)} G(a - c) = 0$. This holds only if $\beta = 0$, as $G(a - c)$ is a nonzero vector. Notice that if $G$ is an empty network, $p^*$ obviously does not depend on $\delta$.

For the third part, $\frac{\partial p_i^*}{\partial a_j} = -\frac{\beta}{2(2 - \beta)} m_{ij}(G, \frac{2\delta}{2 - \beta})$. Since $g_{ij} > 0, \delta > 0$, we have $m_{ij}(G, \frac{2\delta}{2 - \beta}) > 0$ as well. As a result, if $p_i^*$ is independent of $a_j$, $\beta$ must be zero. Similar arguments can be used for the case when $p_i^*$ is independent of $c_j$.

**Proof of Proposition 2** The proof is composed of five parts.

**Part 1:**

By (9),

$$p^* = \frac{a + c}{2} - \frac{\beta}{2}[(2 - \beta)I_n - 2\delta G]^{-1}(a - c).$$

Differentiating with respect to $a$ yields

$$\frac{\partial p^*}{\partial a} = \frac{1}{2} I_n - \frac{\beta}{2}[(2 - \beta)I_n - 2\delta G]^{-1} = \frac{1}{2} I_n - \frac{\beta}{2(2 - \beta)} M(G, \frac{2\delta}{2 - \beta}).$$

The above equation implies that

$$\frac{\partial p_i^*}{\partial a_j} = \Omega_{ij} = \frac{1}{2}id_{i=j} - \frac{\beta}{2(2 - \beta)} m_{ij}(G, \frac{2\delta}{2 - \beta})$$

$$= \begin{cases} 
\frac{1}{2} - \frac{\beta}{2(2 - \beta)} m_{ii}(G, \frac{2\delta}{2 - \beta}) & i = j; \\
-\frac{\beta}{2(2 - \beta)} m_{ij}(G, \frac{2\delta}{2 - \beta}) & i \neq j.
\end{cases}$$

The off-diagonal entries are negative, i.e., $\frac{\partial p_i^*}{\partial a_j} < 0$ if $i \neq j$, and the diagonal entry $\Omega_{ii}^1 = \frac{1}{2} - \frac{\beta}{2(2 - \beta)} m_{ii}(G, \frac{2\delta}{2 - \beta}) < 1/2$ for any $i$.

Recall that the eigenvalues of matrix $\Omega^1$ are

$$\frac{1}{2} - \frac{\beta}{2(2 - \beta - 2\delta \lambda_i(G))} = \frac{1 - \beta - \delta \lambda_i(G)}{(2 - \beta - 2\delta \lambda_i(G))},$$

where $\lambda_i(G), i = 1, 2, \ldots, n$ are eigenvalues of $G$. By Assumption 1

$$(1 - \beta - \delta \lambda_i(G)) > 0, \text{ and } (2 - \beta - 2\delta \lambda_i(G)) = \beta + 2(1 - \beta - \delta \lambda_i(G)) > 0, \forall i.$$

As a consequence, all the eigenvalues of matrix $\Omega^1$ are positive. Therefore, $\Omega^1$ is a positive definite matrix, and all the diagonal entries are strictly positive, i.e., $\frac{\partial p_i^*}{\partial a_i} = \Omega_{ii}^1 > 0$.

Thus, we have

$$\frac{\partial p_i^*}{\partial a_i} = \Omega_{ii}^1 = \frac{1}{2} - \frac{\beta}{2(2 - \beta)} m_{ii}(G, \frac{2\delta}{2 - \beta}) \in (0, 1/2).$$ (28)
Moreover for \(i \neq j\), by positive definiteness of \(\Omega^1\), \(|\Omega^1_{ij}| < \sqrt{\Omega^1_{ii} \Omega^1_{jj}} < \sqrt{1/2 \cdot 1/2} = 1/2\). Therefore,

\[
\frac{\partial p^*_i}{\partial a_j} = \Omega^1_{ij} = -\frac{\beta}{2(2 - \beta)} m_{ii}(G, \frac{2\delta}{2 - \beta}) \in (-1/2, 0),
\]

for \(i \neq j\).

**Part 2:**

By (9), we have

\[
p^* = \frac{a + c}{2} - \frac{\beta}{2} [(2 - \beta)I_n - 2\delta G]^{-1}(a - c).
\]

Differentiating with respect to \(c\) yields

\[
\frac{\partial p^*_i}{\partial c} = \frac{1}{2} I_n + \frac{\beta}{2} \left[ (2 - \beta)I_n - 2\delta G \right]^{-1} \frac{\partial}{\partial c} = \frac{1}{2} I_n + \frac{\beta}{2(2 - \beta)} M(G, \frac{2\delta}{2 - \beta}).
\]

Component-wise, we can rewrite the above as:

\[
\frac{\partial p^*_i}{\partial c} = \begin{cases} \Omega^2_{ii} = \frac{1}{2} I_n + \frac{\beta}{2(2 - \beta)} m_{ii}(G, \frac{2\delta}{2 - \beta}) & i = j; \\ \frac{\beta}{2(2 - \beta)} m_{ij}(G, \frac{2\delta}{2 - \beta}) & i \neq j; \end{cases}
\]

Clearly, every entry of \(\Omega^2\) is positive. More specifically, for \(i = j\),

\[
\frac{\partial p^*_i}{\partial c} = \Omega^2_{ii} = \frac{1}{2} + \frac{\beta}{2(2 - \beta)} m_{ii}(G, \frac{2\delta}{2 - \beta}) = 1 - \Omega^1_{ii} \in (1/2, 1),
\]

by (28). For \(i \neq j\), we obtain

\[
\frac{\partial p^*_i}{\partial c} = \Omega^2_{ij} = \frac{\beta}{2(2 - \beta)} m_{ij}(G, \frac{2\delta}{2 - \beta}) = -\Omega^1_{ij} \in (0, 1/2),
\]

by (29).

**Part 3:**

Differentiating with respect to \(\delta\) in (9), we obtain

\[
\frac{\partial p^*_i}{\partial \delta} = -\beta [(2 - \beta)I_n - 2\delta G]^{-1} G(a - c).
\]

Recall that \(\beta > 0\), \([(2 - \beta)I_n - 2\delta G]^{-1} \succeq 0, G \succeq 0, (a - c) \succeq 0\). Therefore, \(\frac{\partial p^*_i}{\partial \delta} \preceq 0\), i.e., \(\frac{\partial p^*_i}{\partial \delta} \preceq 0\) for any \(i\).

**Part 4:**
Differentiating with respect to $\delta$ in (9) yields
\[
\frac{\partial p^*}{\partial \beta} = -\frac{1}{2}[(2 - \beta)I_n - 2\delta G]^{-1}(a - c) - \frac{\beta}{2}[(2 - \beta)I_n - 2\delta G]^{-2}(a - c).
\]
Clearly $\frac{\partial p^*}{\partial \beta} \preceq 0$ because $\beta > 0$, $[(2 - \beta)I_n - 2\delta G]^{-1} \succeq 0$, $[(2 - \beta)I_n - 2\delta G]^{-2} \succeq 0$, $(a - c) \succeq 0$.

**Part 5:**
Since $G' \succ G$, we have
\[
[(2 - \beta)I_n - 2\delta G']^{-1} - [(2 - \beta)I_n - 2\delta G]^{-1} \\
\geq 0.
\]
As a consequence, by (9),
\[
p^*(G') - p^*(G) = \left\{ \frac{a + c}{2} - \frac{\beta}{2}[(2 - \beta)I_n - 2\delta G']^{-1}(a - c) \right\} - \left\{ \frac{a + c}{2} - \frac{\beta}{2}[(2 - \beta)I_n - 2\delta G]^{-1}(a - c) \right\} = -\frac{\beta}{2} [(2 - \beta)I_n - 2\delta G']^{-1} - [(2 - \beta)I_n - 2\delta G]^{-1}(a - c) \preceq 0.
\]
In other words, $p^*(G') \preceq p^*(G)$. 

**Proof of Theorem 2** By (9), we obtain
\[
p^* = \frac{a + c}{2} - \frac{\beta}{2}[(2 - \beta)I_n - 2\delta G]^{-1}(a - c).
\]
For small $\delta$, the inverse matrix can be expanded as follows:
\[
[(2 - \beta)I_n - 2\delta G]^{-1} = \frac{1}{2 - \beta}I_n - \frac{2\delta}{(2 - \beta)^2}G + O(\delta^2).
\]
Hence,
\[
p^* = \frac{a + c}{2} - \frac{\beta}{2} \left( \frac{1}{2 - \beta}I_n + \frac{2\delta}{(2 - \beta)^2}G + O(\delta^2) \right) (a - c) = \frac{(1 - \beta)a + c}{2 - \beta} + \delta \left( \frac{-\beta}{(2 - \beta)^2}G(a - c) \right) + O(\delta^2).
\]
Specifically, for each customer $i$, we have
\[
p_i^A = p_i^B = \frac{(1 - \beta)a_i + c_i}{2 - \beta} - \frac{\delta\beta}{(2 - \beta)^2} \sum_j g_{ij}(a_j - c_j) + O(\delta^2).
\]
When \( a_i = a, c_i = c \) for all \( i \), we can simplify the above term further:

\[
p_i^A = p_i^B = \frac{(1 - \beta)a + c}{2 - \beta} - \frac{\delta \beta(a - c)}{(2 - \beta)^2}d_i + \mathcal{O}(\delta^2).
\]

\[\square\]

**Proof of the results in Section 5.4.3** For \( K_{pq} \), the adjacency matrix is

\[
G = \begin{bmatrix} 0 & J_{pq} \\ J_{qp} & 0 \end{bmatrix}.
\]

The inverse matrix is

\[
[(2 - \beta)I_n - 2\delta G]^{-1} = \frac{1}{2 - \beta} \begin{bmatrix} I_p + \frac{4\delta^2 q}{(2 - \beta)^2 - 4\delta^2 qp} J_{pp} & \frac{2\delta(2 - \beta)}{(2 - \beta)^2 - 4\delta^2 qp} J_{pq} \\ \frac{2\delta(2 - \beta)}{(2 - \beta)^2 - 4\delta^2 qp} J_{qp} & I_q + \frac{4\delta^2 p}{(2 - \beta)^2 - 4\delta^2 qp} J_{qq} \end{bmatrix}.
\]

By (9), we obtain

\[
p^* = \frac{a + c}{2} - \frac{\beta}{2} \frac{1}{2 - \beta} \begin{bmatrix} I_p + \frac{4\delta^2 q}{(2 - \beta)^2 - 4\delta^2 qp} J_{pp} & \frac{2\delta(2 - \beta)}{(2 - \beta)^2 - 4\delta^2 qp} J_{pq} \\ \frac{2\delta(2 - \beta)}{(2 - \beta)^2 - 4\delta^2 qp} J_{qp} & I_q + \frac{4\delta^2 p}{(2 - \beta)^2 - 4\delta^2 qp} J_{qq} \end{bmatrix} (a - c)
\]

Moreover, under Assumption 3 \((a_i = a, c_i = c)\), they can be further simplified as:

\[
p^* = \frac{(1 - \beta)a + c}{2 - \beta} 1_n - \frac{\beta(a - c)}{(2 - \beta)((2 - \beta)^2 - 4\delta^2 qp)} \begin{bmatrix} (2\delta^2 pq + \delta(2 - \beta)q) 1_p \\ (2\delta^2 pq + \delta(2 - \beta)p) 1_q \end{bmatrix}.
\]

\[\square\]

**Proof of Theorem 3** In equilibrium, according to Theorem 1, the price vector is

\[
p^* = [(2 - \beta)I_n - 2\delta G]^{-1} [(1 - \beta)I_n - \delta G]a + (I_n - \delta G)c.
\]

Therefore,

\[
\begin{align*}
\{ p^* - c & = [(2 - \beta)I_n - 2\delta G]^{-1}[(1 - \beta)I_n - \delta G](a - c). \\
(a - p^* & = [(2 - \beta)I_n - 2\delta G]^{-1}(I_n - \delta G)(a - c).
\end{align*}
\]

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By Corollary [1]

\[ x^* = [(1 + \beta)I_n - \delta G]^{-1}(a - p^*) \]
\[ = [(1 + \beta)I_n - \delta G]^{-1}[(2 - \beta)I_n - 2\delta G]^{-1}[I_n - \delta G](a - c). \]

As a result,

\[ \Pi = \langle x^*, p^* - c \rangle \]
\[ = \langle[(1 + \beta)I_n - \delta G]^{-1}[(2 - \beta)I_n - 2\delta G]^{-1}[I_n - \delta G](a - c), [(1 + \beta)I_n - \delta G]^{-1}[(2 - \beta)I_n - 2\delta G]^{-1}[(1 - \beta)I_n - \delta G](a - c) \rangle \]
\[ = \langle(a - c), [(1 + \beta)I_n - \delta G]^{-1}[(2 - \beta)I_n - 2\delta G]^{-1}[I_n - \delta G][(1 - \beta)I_n - \delta G](a - c) \rangle. \]

**Proof of Proposition 4** Recall that the parameters \( \beta, z \) lie in the domain \( D \). Therefore,

\[ 1 - \delta \lambda_i > 0, \quad 1 - \beta - \delta \lambda_i > 0, \quad 1 + \beta - \delta \lambda_i > 0, \quad 2 - \beta - 2\delta \lambda_i > 0, \quad (30) \]

for any \( \lambda_i \) that is an eigenvalue of \( G \), i.e., for \( \lambda_i \in \text{Spec}(G) \). As a result, all the eigenvalues of \( \phi(G) \), which are just \( \phi(\lambda_1), \ldots, \phi(\lambda_n) \), are positive numbers. Therefore, \( \phi(G) \) is a positive definite square matrix, and accordingly \( \Pi = \langle(a - c), \phi(G)(a - c) \rangle > 0 \) for any \( a \) and \( c \).

Similarly, one can show that

\[ \frac{\partial \Pi}{\partial \beta} = \langle(a - c), \frac{\partial \phi}{\partial \beta}(G)(a - c) \rangle \]

where

\[ \frac{\partial \phi(z)}{\partial \beta} = \frac{-2(1 - \delta z)((1 - \delta z)^2 - (1 - \delta z)\beta + \beta^2)}{(1 + \beta - \delta z)^2(2 - \beta - 2\delta z)^3}. \]

Note that \((1 - \delta z)^2 - (1 - \delta z)\beta + \beta^2 = (1 - \delta z - \frac{1}{2}\beta)^2 + \frac{3}{4}\beta^2 > 0\). Consequently, \( \frac{\partial \phi(z)}{\partial \beta} \big|_{z=\lambda_i} < 0 \) for \( i \) by (30). As a consequence, the eigenvalues of \( \frac{\partial \phi}{\partial \beta}(G) \), which are just \( \frac{\partial \phi}{\partial \beta}(\lambda_i), i = 1, \ldots, n \), are all strictly negative. Therefore, \( \frac{\partial \phi}{\partial \beta}(G) \) is negative definite. Hence, \( \frac{\partial \Pi}{\partial \beta} = \langle(a - c), \frac{\partial \phi}{\partial \beta}(G)(a - c) \rangle < 0 \).

Last, we have

\[ \frac{\partial \Pi}{\partial \delta} = \langle(a - c), \frac{\partial \phi}{\partial \delta}(G)(a - c) \rangle \]

where

\[ \frac{\partial \phi(z)}{\partial \delta} = \frac{-z(\beta^3 + 3(1 - \delta z)\beta - (1 - \delta z)^3)}{(1 + \beta - \delta z)^2(2 - \beta - 2\delta z)^3}. \]

The sign of \( \frac{\partial \phi(z)}{\partial \delta} \big|_{z=\lambda_i} \) equals that of \(-z(\beta^3 + 3(1 - \delta z)\beta - (1 - \delta z)^3) \big|_{z=\lambda_i} \), which has ambiguous sign. Therefore the sign of \( \frac{\partial \Pi}{\partial \delta} \) could be positive or negative, depending on the parameters of the model \( G, \delta, \beta \). For example when \( \delta = 0 \), we have \( \frac{\partial \phi(z)}{\partial \delta} \big|_{\delta=0} = \frac{(2 - 3\beta - \beta^3)}{(1 + \beta)^2(2 - \beta)^3}z \). Therefore, \( \frac{\partial \phi}{\partial \delta}(G) \big|_{\delta=0} = \frac{(2 - 3\beta - \beta^3)}{(1 + \beta)^2(2 - \beta)^3}z \).
\[
\frac{(2-3\beta-\beta^3)}{(1+\beta)^2(2-\beta)^3} G, \text{ and we obtain}
\]
\[
\frac{\partial \Pi}{\partial \delta}|_{\delta=0} = \langle (a-c), \frac{\partial \phi}{\partial \delta}(G)|_{\delta=0}(a-c) \rangle = \frac{(2-3\beta-\beta^3)}{(1+\beta)^2(2-\beta)^3} \langle (a-c), G(a-c) \rangle.
\]

The above equation can be positive (for example when \( \beta = 0.1 \)) or negative (for example when \( \beta = 0.7 \)). Therefore the sign of \( \frac{\partial \Pi}{\partial \delta} \) could be positive or negative. \( \square \)

**Proof of Proposition 6:** By (16), we obtain \( \phi(z)|_{\delta=0} = \frac{(1-\beta)}{(1+\beta)^2(2-\beta)^2} \). Moreover, \( \frac{\partial \phi(z)}{\partial \delta}|_{\delta=0} = \frac{(2-3\beta-\beta^3)}{(1+\beta)^2(2-\beta)^3} z \) by (31). Therefore,
\[
\phi(z) = \frac{(1-\beta)}{(1+\beta)(2-\beta)^2} + (2-3\beta-\beta^3) \frac{(1+\beta)^2(2-\beta)^3}{(1+\beta)^2(2-\beta)^3} z + O(\delta^2).
\]

Hence, we obtain
\[
\phi(G) = \frac{(1-\beta)}{(1+\beta)(2-\beta)^2} I_n + \frac{(2-3\beta-\beta^3)}{(1+\beta)^2(2-\beta)^3} G + O(\delta^2).
\]

As a consequence, we have the following Taylor expansion for the equilibrium profit for small \( \delta \):
\[
\Pi = \langle (a-c), \phi(G)(a-c) \rangle
\]
\[
= \frac{(1-\beta)}{(1+\beta)(2-\beta)^2} \langle (a-c), (a-c) \rangle + \delta \frac{(2-3\beta-\beta^3)}{(1+\beta)^2(2-\beta)^3} \langle (a-c), G(a-c) \rangle + O(\delta^2).
\]

Now we proceed to establish the statement of the proposition. From the above Taylor expansions, we find that
\[
\Pi(G') - \Pi(G) = \delta \frac{(2-3\beta-\beta^3)}{(1+\beta)^2(2-\beta)^3} \langle (a-c), (G' - G)(a-c) \rangle + O(\delta^2).
\]

Observe that \( \langle (a-c), (G' - G)(a-c) \rangle > 0 \) as \( G' > G \) by assumption. The cubic polynomial \( 2-3\beta-\beta^3 \) is decreasing in \( \beta \) and it has a unique real root at \( \beta^* = \sqrt{1+\sqrt{2}} - \frac{1}{\sqrt{1+\sqrt{2}}} \approx 0.596072 \).

Therefore, \( 2-3\beta-\beta^3 > 0 \) if and only if \( \beta < \beta^* \). This cutoff value \( \beta^* \) determines the sign of \( \frac{\partial \Pi}{\partial \delta} \) around \( \delta = 0 \). In other words, for a fixed \( \beta > 0 \), the sign of \( \Pi(G') - \Pi(G) \) is solely determined by the sign of \( \delta \) for \( \delta > 0 \) but close to 0.

Now suppose that we pick any number, say \( \beta_1 = 0.4 < \beta^* \). In this case, we can find a \( \delta_1 > 0 \) small enough such that \( \Pi(G') - \Pi(G)|_{(\beta_1, \delta_1)} > 0 \). Similarly, if we pick another number, say \( \beta_2 = 0.7 > \beta^* \), we can find \( \delta_2 \) small enough such that \( \Pi(G') - \Pi(G)|_{(\beta_2, \delta_2)} > 0 \). The rest just follows from the continuity argument. The proposition is therefore valid. \( \square \)

**Proof of Theorem 3:** First, we present the consumption equilibrium for fixed prices by the firms.
Let
\[ x^t = \begin{bmatrix} x^t_1 \\ \vdots \\ x^t_n \end{bmatrix}, a^t = \begin{bmatrix} a^t_1 \\ \vdots \\ a^t_n \end{bmatrix}, p^t = \begin{bmatrix} p^t_1 \\ \vdots \\ p^t_n \end{bmatrix}, \]
denote the consumption profiles, the marginal utilities, and prices for product \( t \). We further arrange them as follows:
\[ X = \begin{bmatrix} x^1 \\ \vdots \\ x^l \\ x^{l+1} \\ \vdots \\ x^n \end{bmatrix}, A = \begin{bmatrix} a^1 \\ \vdots \\ a^l \\ a^{l+1} \\ \vdots \\ a^n \end{bmatrix}, P = \begin{bmatrix} p^1 \\ \vdots \\ p^l \\ p^{l+1} \\ \vdots \\ p^n \end{bmatrix}, \]
We will need the following lemma.

**Lemma 2.** With multiple products, suppose Assumption \( \Box \) holds. There exists a unique Nash equilibrium in which
\[ X = \begin{bmatrix} x^1 \\ \vdots \\ x^l \\ x^{l+1} \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} W \Phi \cdots \Phi \\ \Phi W \cdots \Phi \\ \vdots \\ \vdots \\ \Phi \cdots \Phi W \end{bmatrix} \begin{bmatrix} a^1 - p^1 \\ \vdots \\ a^l - p^l \end{bmatrix}, \]
where
\[ W = \frac{[I - (1 - \beta)I_n - \delta G]^{-1} + (l - 1)((1 - \beta)I_n - \delta G]^{-1}}{l}, \]
\[ \Phi = \frac{[(1 - \beta)I_n - \delta G]^{-1} - [(1 - \beta)I_n - \delta G]^{-1}}{l}. \]
In particular, when \( a^t = a, p^t = p \) for all \( t \), we have \( x^t = x = (1 + (l - 1)\beta)I_n - \delta G]^{-1}(a - p) \) for all \( t \).

This lemma is directed adapted from [Chen et al. (2015)](http://example.com). Now we study the pricing decision of firms. Given the symmetry assumptions on the marginal utilities and marginal costs, there is a unique symmetric pricing equilibrium in which all firms charge the same price \( p^* \). Analogous to the proofs of Theorem 1 at the symmetric equilibrium prices \( p^* \), if a firm, say firm 1 deviates and lower his price vector by \( \Delta p^1 \), it must satisfy the following no-deviating condition:
\[
\begin{aligned}
\langle \Delta p^1, [(1 + (l - 1)\beta)I_n - \delta G]^{-1}(a - p^*) \rangle &= x^* \\
\Delta x^1 &= \langle \frac{[(1 + (l - 1)\beta)I_n - \delta G]^{-1} + (l - 1)((1 - \beta)I_n - \delta G]^{-1}}{l} \Delta p^1, (p^* - c) \rangle.
\end{aligned}
\]
for any $\Delta p^1$ in $\mathbb{R}^n$. Here we have $\Delta x^1 = \frac{[(1+(l-1)\beta)I_n-\delta G]^{-1}+(l-1)(1-\beta)I_n-\delta G]^{-1}}{l} \Delta p^1$ by Lemma 2.

Therefore, we obtain the following identity:

$$[(1+(l-1)\beta)I_n-\delta G]^{-1}(a-p^*) = \frac{[(1+(l-1)\beta)I_n-\delta G]^{-1}+(l-1)(1-\beta)I_n-\delta G]^{-1}}{l}(p^*-c).$$

Solving this yields

$$p^* = [(2+(l-3)\beta)I_n-2\delta G]^{-1}[(1-\beta)I_n-\delta G]a + [(1+(l-2)\beta)I_n-\delta G]c,$$

which can be rearranged as in the theorem. In this equilibrium, for each firm the consumption vector is

$$x^* = [(1+(l-1)\beta)I_n-\delta G]^{-1}(a-p^*)$$

and the equilibrium profit is $\Pi = \langle x^*, (p^*-c) \rangle$, which can be simplified to $\langle (a-c), \Phi(G)(a-c) \rangle$. This finished the proof of Theorem 3.

**Proof of Proposition 7**: Clearly, when $\beta = 0$, $p^* = \frac{a+c}{2}$ for any $l$. Therefore, the limit is also $\frac{a+c}{2}$. When $\beta > 0$, the second term in (20) is

$$\lim_{l \to +\infty} \Pi = \lim_{l \to +\infty} \frac{2\delta}{2(2+(l-3)\beta)}b(G, \frac{2\delta}{2+(l-3)\beta}, (a-c)) = \frac{1}{2}b(G, 0, (a-c)) = \frac{(a-c)}{2}.$$

Thus, $\lim_{l \to +\infty} p^* = \frac{a+c}{2} - \frac{a-c}{2} = c$.

Now we turn to the profits. When $\beta > 0$, in the limit the prices converge to the marginal costs; therefore, each firm earns zero profit. When $\beta = 0$, the equilibrium price is $p^* = \frac{a+c}{2}$ for every $l$, and the consumption vector is $x^* = [I_n-\delta G]^{-1}(a-p^*) = \frac{1}{2}[I_n-\delta G]^{-1}(a-c)$. Hence, the profit is $\Pi = \langle x^*, (p^*-c) \rangle = \frac{1}{2}((a-c), [I_n-\delta G]^{-1}(a-c))$.

**Proof of Theorem 4**: Let $p^{A^*} = p^{B^*} = p^u 1_n$ be the unique symmetry uniform price. Now suppose firm $A$ unilaterally deviates and decreases his price by $\Delta p^{A^*} = \Delta p^A 1_n$ (lowers the price by $\Delta p^A$ for every customer), i.e., $p^{A^*} \sim p^* - \Delta p^{A^*}$. It has two effects. On one hand, the price margins per unit are lower. This gives rise to the total marginal loss

$$\approx \langle x^*, \Delta p^{A^*} \rangle = \langle M^+(a-p^u 1_n), \Delta p^{A^*} \rangle = \langle M^+(a-p^u 1_n), \Delta p^A 1_n \rangle.$$

On the other hand, there is marginal benefit due to demand enhancing, which is $\approx (\Delta x^A, p^u 1_n - c)$. The change in consumption for $A$ due to lower $p^{A^*}$ is $\Delta x^A = \frac{M^++M^-}{2} \Delta p^{A^*} = \frac{M^++M^-}{2} \Delta p^A 1_n$. In equilibrium both effects must cancel out, i.e.,

$$\langle M^+(a-p^u 1_n), \Delta p^A 1_n \rangle = \langle \frac{M^++M^-}{2} \Delta p^A 1_n, p^u 1_n - c \rangle,$$

for any $\Delta p^A \in \mathbb{R}$. 43
As a consequence,
\[ \langle \mathbf{M}^+(a-p^u1_n), 1_n \rangle = \langle \frac{\mathbf{M}^+ + \mathbf{M}^-}{2}1_n, p^u1_n - c \rangle. \]
Plugging in \( \mathbf{M}^+ \) and \( \mathbf{M}^- \) and simplifying yields the theorem. \( \square \)

**Proof of Theorem 5** First, we present the consumption equilibrium as a lemma.

**Lemma 3.** Assume the following stability condition:
\[ (1 + \beta) - |\delta + \mu| \lambda_1(\mathbf{G}) > 0, \quad \text{and} \quad (1 - \beta) - |\delta - \mu| \lambda_1(\mathbf{G}) > 0. \]
Then there exists a unique equilibrium in the consumption stage with
\[ \begin{bmatrix} x^A \\ x^B \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{M}}^+ \mathbf{M}^- - \frac{1}{2} \tilde{\mathbf{M}}^+ \mathbf{M}^- \\ \mathbf{M}^+ - \frac{1}{2} \mathbf{M}^- \end{bmatrix} \begin{bmatrix} a^A - p^A \\ a^B - p^B \end{bmatrix}, \]
where matrices \( \tilde{\mathbf{M}}^+ \) and \( \tilde{\mathbf{M}}^- \) are defined as follows.
\[ \tilde{\mathbf{M}}^+ = [(1 + \beta)\mathbf{I}_n - (\delta + \mu)\mathbf{G}]^{-1} = \sum_{k \geq 0} \frac{(\delta + \mu)^k \mathbf{G}^k}{(1 + \beta)^{1+k}}, \]
\[ \tilde{\mathbf{M}}^- = [(1 - \beta)\mathbf{I}_n - (\delta - \mu)\mathbf{G}]^{-1} = \sum_{k \geq 0} \frac{(\delta - \mu)^k \mathbf{G}^k}{(1 - \beta)^{1+k}}. \]
Moreover, when \( a^A = a^B = a \), and \( p^A = p^B = p^* \), then \( a^A = a^B = x = \tilde{\mathbf{M}}^+(a - p^*) \).

This lemma is directly adapted form Chen et al. (2015). Following the similar steps as the derivations of Theorem 1, we can show that a symmetric equilibrium \( p^A = p^B = p^* \) is pinned down by the following condition:
\[ \tilde{\mathbf{M}}^+(a - p^*) = \frac{1}{2}(\tilde{\mathbf{M}}^+ + \tilde{\mathbf{M}}^-)(p^* - c). \]
Plugging in \( \tilde{\mathbf{M}}^+ \) and \( \tilde{\mathbf{M}}^- \) and simplifying yields
\[ p^* = (3\tilde{\mathbf{M}}^+ + \tilde{\mathbf{M}}^-)^{-1} \left\{ 2\tilde{\mathbf{M}}^+ a + (\tilde{\mathbf{M}}^+ + \tilde{\mathbf{M}}^-)c \right\} = [(2 - \beta)\mathbf{I}_n - (2\delta - \mu)\mathbf{G}]^{-1} [(1 - \beta)\mathbf{I}_n - (\delta - \mu)\mathbf{G})a + (\mathbf{I}_n - \delta \mathbf{G})c], \]
which finished the proof of Theorem 5. \( \square \)

**Proof of Proposition 8** From (23), we obtain
\[ p^* = [(2 - \beta)\mathbf{I}_n - (2\delta - \mu)\mathbf{G}]^{-1} [(1 - \beta)\mathbf{I}_n - (\delta - \mu)\mathbf{G})a + (\mathbf{I}_n - \delta \mathbf{G})c] = \mathbf{c} + [(2 - \beta)\mathbf{I}_n - (2\delta - \mu)\mathbf{G}]^{-1} [(1 - \beta)\mathbf{I}_n - (\delta - \mu)\mathbf{G})a - c] \]
Differentiating with respect to $\mu$ yields

$$\frac{\partial \Pi^c}{\partial \mu} = [(2 - \beta)I_n - (2\delta - \mu)G]^{-2}(1 - \delta)G(a - c).$$

Notice that $2\delta - \mu = \delta + (\delta - \mu) > 0$, and so $[(2 - \beta)I_n - (2\delta - \mu)G]^{-1} = \sum_{k \geq 0}^{\infty}(2\delta - \mu)^kG^k \geq 0$. Therefore, $[(2 - \beta)I_n - (2\delta - \mu)G]^{-2} \succeq 0$ as well. Hence, $\frac{\partial \Pi^c}{\partial \mu} \succeq 0$. When $\mu = 0$, we obtain the price $p^* \succeq p^c$.

**Proof of Theorem 6** In equilibrium, the following must hold:

$$\frac{\partial \Pi^A}{\partial p^A} \bigg|_{p^A = \hat{p}^A} = 0, \text{ and } \frac{\partial \Pi^B}{\partial p^B} \bigg|_{p^B = \hat{p}^B} = 0.$$ 

From (6) and (7), these first-order conditions can be written as

\[
\begin{align*}
\begin{bmatrix}
\frac{M^+ + M^-}{2} & \frac{M^+ - M^-}{2} \\
\frac{M^+ - M^-}{2} & \frac{M^+ + M^-}{2}
\end{bmatrix}
\begin{bmatrix}
a^A - \hat{p}^A \\
a^B - \hat{p}^B
\end{bmatrix}
&= \begin{bmatrix}
\frac{M^+ + M^-}{2} \\
0
\end{bmatrix}\begin{bmatrix}
\hat{p}^A - c^A \\
\hat{p}^B - c^B
\end{bmatrix}.
\end{align*}
\]

(33)

Taking the summation in (33) yields

$$M^+(a^A + a^B - \hat{p}^A - \hat{p}^B) = \frac{M^+ + M^-}{2}(\hat{p}^A + \hat{p}^B - c^A - c^B).$$

Therefore,

$$\hat{p}^A + \hat{p}^B = (3M^+ + M^-)^{-1}\{2M^+(a^A + a^B) + (M^+ + M^-)(c^A + c^B)\}. $$

Plugging in $M^+ = [(1+\beta)I_n - \delta G]^{-1}$, $M^- = [(1 - \beta)I_n - \delta G]^{-1}$, and simplifying it, we obtain that

$$\frac{\hat{p}^A + \hat{p}^B}{2} = \frac{a^A + a^B}{2} + \frac{c^A + c^B}{2} - \frac{\beta}{2(2-\beta)} b \left(G, \frac{2\delta}{2 - \beta}, \left(\frac{a^A + a^B}{2} - \frac{c^A + c^B}{2}\right)\right).$$

Similarly, taking the difference in (33) yields

$$M^-(a^A - a^B - \hat{p}^A + \hat{p}^B) = \frac{M^+ + M^-}{2}(\hat{p}^A - \hat{p}^B - c^A + c^B).$$
and therefore

\[ \hat{p}^A - \hat{p}^B = (3M^- + M^+)^{-1} \{ 2M^- (a^A - a^B) + (M^+ + M^-) (c^A - c^B) \} . \]

Plugging in \( M^+ = [(1 + \beta)I_n - \delta G]^{-1} \), \( M^- = [(1 - \beta)I_n - \delta G]^{-1} \) and simplifying it, the above equation can be rewritten as

\[ \frac{\hat{p}^A - \hat{p}^B}{2} = [(2 + \beta)I_n - 2\delta G]^{-1} \left[ (1 + \beta)I_n - \delta G \right] \frac{a^A - a^B}{2} + (I_n - \delta G) \frac{c^A - c^B}{2} \]  
\[ = \frac{a^A - a^B}{2} + \frac{c^A - c^B}{2} + \frac{\beta}{2(2 + \beta)} b \left( G, \frac{a^A - a^B}{2} - \frac{c^A - c^B}{2} \right) \]  
\[ \text{Combing results in (34) and (35) yields} \]

\[
\begin{cases}
\hat{p}^A = \frac{a^A + c^A}{2} - \frac{\beta}{2(2 + \beta)} b \left( G, \frac{a^A + c^A}{2} \right) + \frac{\beta}{2(2 + \beta)} b \left( G, \frac{a^A - c^A}{2} \right), \\
\hat{p}^B = \frac{a^B + c^B}{2} - \frac{\beta}{2(2 + \beta)} b \left( G, \frac{a^B + c^B}{2} \right) - \frac{\beta}{2(2 + \beta)} b \left( G, \frac{a^B - c^B}{2} \right).
\end{cases}
\]

Note that for the special case with \( a^A = a^B = a \) and \( c^A = c^B = c \), we must have \( \hat{p}^A = \hat{p}^B \) by (35). Therefore, the equilibrium price is symmetric. Moreover, this common price vector equals

\[ [(2 - \beta)I_n - 2\delta G]^{-1} \frac{a + c}{2} + \frac{\beta}{2} [(2 - \beta)I_n - 2\delta G]^{-1} (a - c) \]

by (34), which is consistent with Theorem 1. Hence, we obtain the result in the theorem. \( \square \)

C Single representative customer case

In this appendix, we consider the case without the network effects. We investigate this benchmark using the single representative customer setup \((n = 1)\). The customer’s utility function is:

\[ u(x^A, x^B) + I = a^A x^A + a^B x^B - \left( \frac{1}{2} (x^A)^2 + (x^B)^2 + \beta x^A x^B \right) + I. \]  
\[ \text{(37)} \]

Here \( I \) is the composite good (i.e., it serves as the numeraire). The parameter \( \beta \) satisfies \(|\beta| < 1\).

Given this utility function, the customer’s problem is

\[ \max_{\{x^A, x^B, I\}} \{ U(x^A, x^B, I) \} \quad \text{s.t.} \quad p^A x^A + p^B x^B + I \leq w, \]

or equivalently,

\[ \max_{\{x^A, x^B\}} \left\{ (a^A - p^A) x^A + (a^B - p^B) x^B - \left( \frac{1}{2} (x^A)^2 + (x^B)^2 + \beta x^A x^B \right) \right\} . \]
Let
\[ \Psi = \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix}, \quad \Psi^{-1} = \frac{1}{1 - \beta^2} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix}, \]
which is consistent with the notation in Section 7.1 except that now there are only two products. The first-order conditions can be written as
\[ \begin{bmatrix} a^A - p^A \\ a^B - p^B \end{bmatrix} = \Psi \begin{bmatrix} x^A \\ x^B \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} x^A \\ x^B \end{bmatrix} = \Psi^{-1} \begin{bmatrix} a^A - p^A \\ a^B - p^B \end{bmatrix} = \frac{1}{1 - \beta^2} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} a^A - p^A \\ a^B - p^B \end{bmatrix}. \quad (38) \]
These are the demand functions for the differentiated Bertrand competition model.

**Monopoly setup.** In this case, a monopoly firm sells both products. Let \( c^A \) and \( c^B \) be the marginal costs of products \( A \) and \( B \). The monopoly firm’s objective is
\[
\max_{\{p^A, p^B\}} \left[ p^A - c^A \right] \left[ p^B - c^B \right] \frac{1}{1 - \beta^2} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} a^A - p^A \\ a^B - p^B \end{bmatrix}.
\]
Let \( \bar{p}^A, \bar{p}^B \) be the solutions. The first-order conditions for the joint maximization lead to
\[ \frac{1}{1 - \beta^2} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} a^A + c^A - 2\bar{p}^A \\ a^B + c^B - 2\bar{p}^B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]
Since \( \Psi^{-1} \) is invertible, we have
\[ \begin{bmatrix} a^A + c^A - 2\bar{p}^A \\ a^B + c^B - 2\bar{p}^B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Rightarrow \quad \begin{bmatrix} \bar{p}^A \\ \bar{p}^B \end{bmatrix} = \begin{bmatrix} a^A + c^A \\ a^B + c^B \end{bmatrix}. \quad (39) \]

**Duopoly setup.** Now suppose that the two prices are controlled by two different firms. Let
\[ \pi^A(p^A, p^B) = (p^A - c^A)x^A = \frac{1}{1 - \beta^2}(p^A - c^A)((a^A - p^A) - \beta(a^B - p^B)), \]
\[ \pi^B(p^A, p^B) = (p^B - c^B)x^B = \frac{1}{1 - \beta^2}(p^B - c^B)((a^B - p^B) - \beta(a^A - p^A)). \]
The first-order conditions for the Nash equilibrium are:
\[ \frac{\partial \pi^A(p^A, p^B)}{\partial p^A} = 0, \quad \Rightarrow \quad ((a^A - p^A) - \beta(a^B - p^B)) = (p^A - c^A), \]
\[ \frac{\partial \pi^B(p^A, p^B)}{\partial p^B} = 0, \quad \Rightarrow \quad ((a^B - p^B) - \beta(a^A - p^A)) = (p^B - c^B). \]
In matrix forms, we obtain that:
\[
\begin{bmatrix}
2 & -\beta \\
-\beta & 2
\end{bmatrix}
\begin{bmatrix}
p^A \\
p^B
\end{bmatrix}
= \begin{bmatrix}
a^A + c^A - \beta a^B \\
a^B + c^B - \beta a^A
\end{bmatrix},
\]
Therefore, the equilibrium prices are
\[
\begin{bmatrix}
p^A^* \\
p^B^*
\end{bmatrix} = \begin{bmatrix}
2 & -\beta \\
-\beta & 2
\end{bmatrix}^{-1}
\begin{bmatrix}
a^A + c^A - \beta a^B \\
a^B + c^B - \beta a^A
\end{bmatrix}
= \frac{1}{4 - \beta^2}
\begin{bmatrix}
2(a^A + c^A - \beta a^B) + \beta(a^B + c^B - \beta a^A) \\
2(a^B + c^B - \beta a^A) + \beta(a^A + c^A - \beta a^B)
\end{bmatrix}.
\]
(40)

When \(a^A = a^A = a, c^A = c^B = c\), the duopoly prices are
\[
\begin{bmatrix}
p^A^* \\
p^B^*
\end{bmatrix} = \begin{bmatrix}
(1 - \beta)a + c \\
(2 - \beta)(1 - \beta)a + c
\end{bmatrix}.
\]

When \(\beta > 0\), for substitute products, the duopoly price \(\frac{(1 - \beta)a + c}{2 - \beta}\) is lower than the monopoly price \(\frac{a^2 + c}{2}\). When \(\beta < 0\), for complements products, the duopoly price \(\frac{(1 - \beta)a + c}{2 - \beta}\) is actually higher than the monopoly price \(\frac{a^2 + c}{2}\). This results have obvious counter-parts for \(l \geq 2\) products.

### D Additional results in the oligopoly model

By Theorem 3, the equilibrium price in the oligopoly model is
\[
p^* = \frac{a + c}{2} - \frac{(l - 1)\beta}{2}[(2 + (l - 3)\beta)I_n - 2\delta G]^{-1}(a - c).
\]

For small \(\delta\),
\[
p^* = \frac{[(1 - \beta)a + (1 + (l - 2)\beta)c]}{(2 + (l - 3)\beta)} - \delta \frac{(l - 1)\beta}{(2 + (l - 3)\beta)^2}G(a - c) + O(\delta^2).
\]

As long as \(\beta \neq 0\) (when the products are not completely independent) and \(l \neq 1\) (there is competition), we know that the equilibrium price will depend on the network structure \(G, a_k, c_k\), etc. As a remark, the results are valid when we set \(l = 1\). In this case, \(p = \frac{a + c}{2}\), and \(x^* = [I_n - \delta G]^{-1} \frac{a - c}{2}\), and \(\Pi = \langle (a - c), \frac{1}{4} [I_n - \delta G]^{-1} (a - c) \rangle\).

Most of the results for duopoly case have parallel counterparts for \(l \geq 2\). For simplicity, we only present the results using regular graphs. For a regular graph with degree \(d\), we assume that
\[ a_i = a, c_i = c, i = 1, \ldots, n. \] Then by Theorem 3, the equilibrium prices are

\[ p^* = \left( \frac{(2 + (l - 3)\beta)I_n - 2\delta G}{1 - \beta - \delta d} a + (1 + (1 - 2)\beta - \delta d) c}{2 + (l - 3)\beta - 2\delta d} \right) I_n. \]

Let

\[ p^d := \frac{(1 - \beta - \delta d) a + (1 + (1 - 2)\beta - \delta d) c}{2 + (l - 3)\beta - 2\delta d} = c + \frac{(1 - \beta - \delta d)}{2 + (l - 3)\beta - 2\delta d} (a - c). \]

Clearly, \( \frac{\partial p^d}{\partial a} > 0 \), and \( \frac{\partial p^d}{\partial c} > 0 \). Moreover, the derivatives can be easily calculated:

\[ \frac{\partial p^d}{\partial \beta} = \frac{-(l - 1)(1 - \delta d)}{(2 + (l - 3)\beta - 2\delta d)^2} (a - c) < 0, \]

\[ \frac{\partial p^d}{\partial \delta} = \frac{-(l - 1)d\beta}{(2 + (l - 3)\beta - 2\delta d)^2} (a - c), \]

\[ \frac{\partial p^d}{\partial d} = \frac{-(l - 1)\delta\beta}{(2 + (l - 3)\beta - 2\delta d)^2} (a - c), \]

\[ \frac{\partial p^d}{\partial l} = \frac{-\beta(1 - \beta - \delta d)}{(2 + (l - 3)\beta - 2\delta d)^2} (a - c). \]

As a result,

\[ \frac{\partial p^d}{\partial \beta} < 0, \quad \text{sign} \left( \frac{\partial p^d}{\partial \delta} \right) = -\text{sign} \{ \beta \}, \quad \text{sign} \left( \frac{\partial p^d}{\partial d} \right) = -\text{sign} \{ \beta \delta \}. \]

When \( \beta = 0 \), \( p^d = \frac{a + c}{2} \) is independent of \( \delta, d, \) and \( l \). This is equivalent to the monopoly case, as \( \beta = 0 \) implies no interactions among different products. In the range where \( \beta > 0, \delta > 0 \), the equilibrium price \( p^d \) is decreasing in \( \delta \), decreasing in \( \beta \), decreasing in the degree \( d \), and decreasing in the number of firms \( l \). Moreover, \( \lim_{l \to \infty} p^d = c \). The price converges to the marginal cost as the number of firms grows large. Recall that when \( \beta = 0 \), \( p^d = \frac{a + c}{2} \) is independent of the number of firms \( l \). This again suggests that the independent product case is a knife-edge scenario.

We can also investigate the firms’ profits. The results parallel those for the duopoly setup and are highlighted below.

**Proposition 9.** Suppose more than two firms compete in the market.

- For any \( G \), \( \Pi \) may be decreasing or increasing in \( \delta \).
- For two graphs \( G' \succ G \), it's possible that the profit under \( G' \) dominates that under \( G \), it is also possible that the opposite occurs.

**Proof of Proposition 9** We can apply Taylor expansions to obtain the results of Proposition 9.
Specifically, we have the following Taylor expansion result for the firms’ profit for small $\delta$:

$$
P = \frac{(1 + (l - 2)\beta)(1 - \beta)}{(1 + (l - 1)\beta)(2 + (l - 3)\beta)^2} \langle (a - c), (a - c) \rangle + \delta^2 \frac{2 + 3(l - 3)\beta - 6(l - 2)\beta^2 - (l^3 - 2l^2 - 2l + 5)\beta^3}{(1 + (l - 1)\beta)^2(2 + (l - 3)\beta)^3} \langle (a - c), G(a - c) \rangle + \mathcal{O}(\delta^2)
$$

Notice that the sign of

$$
2 + 3(l - 3)\beta - 6(l - 2)\beta^2 - (l^3 - 2l^2 - 2l + 5)\beta^3
$$

can be positive or negative for $\beta \in [0, 1)$. When $\beta = 1$, the above term equals $- l(l - 1)^2 < 0$. When $\beta = 0$, the above term equals $2 > 0$. Thus, for any graph $G$, we can have a parameter pair $(\beta, \delta)$ such that $\Pi$ is increasing in $\delta$ at that point, and another pair $(\beta', \delta')$ such that $\Pi$ is increasing in $\delta$ at that point. Similarly, we can show the second part. \hfill \square

For regular graph, under $(FS)$, we have the following expression for equilibrium profit:

$$
P = n(a - c)^2 \Phi(d) = n(a - c)^2 \frac{(1 + (l - 2)\beta - \delta d)(1 - \beta - \delta d)}{(1 + (l - 1)\beta - \delta d)(2 + (l - 3)\beta - 2\delta d)^2}.
$$

Notice that

$$
\frac{\partial \Pi}{\partial \delta} = \frac{dn(a - c)^2 \left[ (-5 - 2l + 2l^2 + l^3)\beta^3 - 6(l - 2)\beta^2(1 - d\delta) + 3(l - 3)\beta(1 - d\delta)^2 + 2(1 - d\delta)^3 \right]}{(1 + (l - 1)\beta - \delta d)^2(2 + (l - 3)\beta - 2\delta d)^3}.
$$

Therefore,

$$
\text{sign} \left( \frac{\partial \Pi}{\partial \delta} |_{\delta = 0} \right) = \text{sign} \left( 2 + 3(l - 3)\beta - 6(l - 2)\beta^2 - (l^3 - 2l^2 - 2l + 5)\beta^3 \right).
$$

Let

$$
g(\beta) := 2 + 3(l - 3)\beta - 6(l - 2)\beta^2 - (l^3 - 2l^2 - 2l + 5)\beta^3.
$$

When $\beta = 1$, $g(1) = - l(l - 1)^2 < 0$. Thus, $\frac{\partial \Pi}{\partial \delta} |_{\delta = 0, \beta = 1} < 0$. When $\beta = 0$, $g(0) = 2 > 0$. This implies that $\frac{\partial \Pi}{\partial \delta} |_{\delta = 0, \beta = 0} > 0$ and consequently it is possible that $\frac{\partial \Pi}{\partial \delta} > 0$ or $\frac{\partial \Pi}{\partial \delta} < 0$. As $l$ increases, $\Pi$ decreases because

$$
\frac{\partial \Pi}{\partial l} = - n(a - c)^2 \beta (1 - \beta - \delta d) \left[ \frac{((2l^2 - 7l + 7)\beta^3 + 4(l - 2)\beta(1 - d\delta) + 2(1 - d\delta)^2)}{(1 + (l - 1)\beta - \delta d)^2(2 + (l - 3)\beta - 2\delta d)^3} \right] < 0,
$$

and $\lim_{l \to \infty} \Pi = 0$ as the price is $\lim_{l \to \infty} p^d = c$. We again notice that the only exception is when $\beta = 0$. In this case, the profit $\Pi$ is independent of $l$, as $p^d = (a + c)/2$ for all $l$. Because of this separable monopoly phenomenon, the equilibrium prices do not converge to the marginal cost.
Similarly, for $\beta$, we have

$$\frac{\partial \Pi}{\partial \beta} = -(l-1)n(a-c)^2 \frac{(2(1 - d\delta)^2 + (4l - 10)(1 - d\delta)^2 \beta + 2(l^2 - 5l + 7)(1 - d\delta)\beta^2 - (6 - 5l + l^2)\beta^3)}{(1 + (l-1)\beta - \delta d)^2(2 + (l-3)\beta - 2\delta d)^3} < 0,$$

in the parameter range we consider.