Key Players under Incomplete Information

Joan de Martí[*] Yves Zenou[†]

October 9, 2015

Abstract

We consider a network game with strategic complementarities where the individual reward is unknown. We study the impact of incomplete information on a network policy which aim is to target the most relevant agents in the network (key players). Compared to the complete information case, we show that the optimal targeting may be very different.

Keywords: Bayesian games, key player policy.
JEL Classification: C72, D82, D85.

[*]Universitat Pompeu Fabra and Barcelona GSE. E-mail: joan.demarti@upf.edu.
[†]Corresponding author: Stockholm University, IFN, Sweden, and CEPR. E-mail: yves.zenou@ne.su.se.

1 Introduction

Network analysis is a growing field within economics[1] because it can analyze situations where agents interact with each other and provides interesting predictions in terms of equilibrium behavior. A recent branch of the network literature has focused on how the *network structure* influences individuals' outcomes. This is modeled by what are sometimes referred to as "games on networks" because the network is assumed to be fixed and the focus is on the game played in the network. An important paper in this literature is that of Ballester et al. (2006). They compute the Nash equilibrium of a network game with *strategic complementarities* when agents choose their efforts simultaneously. In their setup, restricted to linear-quadratic utility functions, they establish that the network game has a unique Nash equilibrium where each agent effort's is proportional to her Katz-Bonacich centrality measure. This is a measure introduced by Katz (1953) and Bonacich (1987), which counts all paths starting from an agent but gives a smaller value to connection that are farther away.

De Martí and Zenou (2016) consider a model similar to that of Ballester et al. (2006) but where the individual reward is *partially known*. In other words, they look at a model where the state of world (i.e. the marginal return of effort) is common to all agents but only partially known by them. They demonstrate that there exists a unique Bayesian-Nash equilibrium and give a complete characterization of equilibrium efforts as a function of weighted Katz-Bonacich centralities.

In the present paper, we study a policy analyzed by Ballester et al. (2006, 2010), the so-called *key-player* policy, in the context of incomplete information. The aim of this policy consists in finding and getting rid of the key player, i.e., the agent who, once removed, leads to the highest reduction in aggregate activity.[2] If the planner has incomplete information about the marginal return of effort, then we show that the key player may be different to the one proposed in the perfect information case. This difference is determined by a ratio that captures all the (imperfect) information that the agents have, including the priors of the agents and the planner and the posteriors of the agents.

[1] For overviews on the network literature, see Jackson (2008), Ioannides (2012) and Jackson and Zenou (2015).

[2] For a recent overview on the literature on key players, see Zenou (2016).

2 Complete information

2.1 The model

The network Let $\mathcal{I} := \{1, \ldots, n\}$ denote the set of players, where $n > 1$, connected by a network $\mathbf{g}$. We keep track of social connections in this network by its symmetric *adjacency matrix* $\mathbf{G} = [g_{ij}]$, where $g_{ij} = g_{ji} = 1$ if i and j are linked to each other, and $g_{ij} = 0$, otherwise. We also set $g_{ii} = 0$.

Payoffs In the perfect information case, each agent takes action $x_i \in [0, +\infty)$ that maximizes the following quadratic utility function:

$$u_i\left(x_i, x_{-i}; \mathbf{G}\right) = \alpha x_i - \frac{1}{2} x_i^2 + \beta \sum_{j=1}^{n} g_{ij} x_i x_j \tag{1}$$

where $\alpha > 0$ is the *marginal return of effort* and $\beta > 0$ is the *strength of strategic interactions*.

2.2 Key players

Consider the utility function defined by (1). Denote by $\lambda_{max}(\mathbf{G})$ the largest eigenvalue of $\mathbf{G}$. Ballester et al. (2006) show that, at the Nash equilibrium, if $\beta < 1/\lambda_{max}(\mathbf{G})$, then each individual i will provide effort $x_i^* = \alpha\, b_i\left(\beta, \mathbf{G}\right)$, where $b_i\left(\beta, \mathbf{G}\right)$ is the Katz-Bonacich centrality of individual i and defined, in vector form, as $\mathbf{b} = \left(\mathbf{I} - \beta \mathbf{G}\right)^{-1} \mathbf{1}$, where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{1}$ is the $n-$vector of 1. Denote by

$$x^*(\mathbf{G}) = \sum_{i=1}^{n} x_i^* = \alpha \sum_{i=1}^{n} b_i\left(\beta, \mathbf{G}\right)$$

the total sum of efforts at the Nash equilibrium. The planner's objective is to generate the highest possible reduction in aggregate effort level by picking the appropriate individual. Formally, the planner's problem is the following:

$$\max_{i \in \{1, \ldots, n\}} \{x^*(\mathbf{G}) - x^*(\mathbf{G}^{-i})\}$$

where $\mathbf{G}^{-i}$ is the $(n-1) \times (n-1)$ adjacency matrix corresponding to the network $\mathbf{g}^{-i}$ when individual i has been removed. From Ballester et al. (2006, 2010), we now define a new network centrality measure, called the *intercentrality* of agent i, and denoted by $IC_i\left(\beta, \mathbf{G}\right)$ that will solve this program. We have the following result:

Proposition 1 (Ballester et al. (2006)) *Assume* $\beta < 1/\lambda_{max}(\mathbf{G})$. *Then, under complete information, the key player in network* $\mathbf{g}$ *is the agent* i *that has the highest intercentrality measure* $IC_i(\beta, \mathbf{G})$*, which is defined as:*

$$IC_i(\beta, \mathbf{G}) := \frac{[b_i(\beta, \mathbf{G})]^2}{m_{ii}(\beta, \mathbf{G})} \tag{2}$$

where $m_{ii}(\beta, \mathbf{G})$ *is the cell corresponding to the* i*th row and the* i*th column of the matrix* $(\mathbf{I}_n - \beta\mathbf{G})^{-1}$ *and thus keeps track of the paths that start and finish at* i *(cycles).*

This proposition says that the key player i^* who solves $\max_{i\in\{1,\ldots,n\}}\{x^*(\mathbf{G}) - x^*(\mathbf{G}^{-i})\}$ is the agent who has the highest inter-centrality $IC_i(\beta, \mathbf{G})$ in $\mathbf{g}$, that is, for all $i = 1, \ldots, n$, $IC_{i^*}(\beta, \mathbf{G}) \geq IC_i(\beta, \mathbf{G})$. As a result,

$$IC_{i^*}(\beta, \mathbf{G}) \in \max_{i\in\{1,\ldots,n\}}\{x^*(\mathbf{G}) - x^*(\mathbf{G}^{-i})\} \tag{3}$$

3 Incomplete information: Key-player policies

We would like now to derive the key-player policy where there is incomplete information. To be more precise, assume that the marginal return of effort α in the payoff function (1) is *common* to all agents but only *partially known* by the agents. Agents know, however, the exact value of the synergy parameter β.

3.1 Bayesian-Nash Equilibrium

We assume that there are *two states of the world*, so that the parameter α can only take two values: $\alpha_l < \alpha_h$. All individuals share a common prior:

$$\mathbb{P}(\{\alpha = \alpha_h\}) = p \in (0, 1) \tag{4}$$

Each individual i receives a private signal, $s_i \in \{h, l\}$, such that

$$\mathbb{P}(\{s_i = h\} \mid \{\alpha = \alpha_h\}) = \mathbb{P}(\{s_i = l\} \mid \{\alpha = \alpha_l\}) = q \geq 1/2$$

where $\{s_i = h\}$ and $\{s_i = l\}$ denote, respectively, the event that agent i has received a signal h and l. Assume that there is no communication between the players and that the network does not affect the possible channels of communication between them. Agent i has to choose an action $x_i(s_i) \geq 0$ for each signal $s_i \in \{l, h\}$. The expected utility of agent i can be written as:

$$\mathbb{E}[u_i|s_i] = \mathbb{E}[\alpha|s_i]\, x_i(s_i) - \frac{1}{2}[x_i(s_i)]^2 + \beta x_i(s_i) \sum_{j=1}^{n} g_{ij} \mathbb{E}[x_j|s_i]$$

We have:

$$\widehat{\alpha}_l := \mathbb{E}[\alpha|\{s_i = l\}] = \frac{q(1-p)}{q(1-p)+(1-q)p}\alpha_l + \frac{(1-q)p}{q(1-p)+(1-q)p}\alpha_h \tag{5}$$

$$\widehat{\alpha}_h := \mathbb{E}_i[\alpha|\{s_i = h\}] = \frac{(1-q)(1-p)}{(1-q)(1-p)+qp}\alpha_l + \frac{qp}{(1-q)(1-p)+qp}\alpha_h \tag{6}$$

$$\gamma_l = \mathbb{P}(\{s_j = l\}\,|\,\{s_i = l\}) = \frac{(1-p)q^2 + p(1-q)^2}{q(1-p)+(1-q)p} \tag{7}$$

$$\gamma_h = \mathbb{P}(\{s_j = h\}\,|\,\{s_i = h\}) = \frac{(1-p)(1-q)^2 + pq^2}{qp+(1-q)(1-p)} \tag{8}$$

De Marti and Zenou (2016) have shown the following result:

Proposition 2 (De Marti and Zenou (2016)) *Consider the network game with payoffs (1) and unknown parameter α that can only take two values: $0 < \alpha_l < \alpha_h$. Then, if $\beta < 1/\lambda_{\max}(\mathbf{G})$, there exists a unique interior Bayesian-Nash equilibrium in pure strategies given by*

$$\underline{\mathbf{x}}^* = \widehat{\alpha}\,\mathbf{b}(\beta, \mathbf{G}) - \frac{(1-\gamma_l)}{(2-\gamma_h-\gamma_l)}(\widehat{\alpha}_h - \widehat{\alpha}_l)\,\mathbf{b}((\gamma_h+\gamma_l-1)\beta, \mathbf{G}) \tag{9}$$

$$\overline{\mathbf{x}}^* = \widehat{\alpha}\,\mathbf{b}(\beta, \mathbf{G}) + \frac{(1-\gamma_h)}{(2-\gamma_h-\gamma_l)}(\widehat{\alpha}_h - \widehat{\alpha}_l)\,\mathbf{b}((\gamma_h+\gamma_l-1)\beta, \mathbf{G}) \tag{10}$$

where

$$\widehat{\alpha} \equiv \frac{(1-\gamma_h)\widehat{\alpha}_l + (1-\gamma_l)\widehat{\alpha}_h}{(2-\gamma_h-\gamma_l)}, \tag{11}$$

γ_l and γ_h are given by (7), and (8) and $\widehat{\alpha}_l$ and $\widehat{\alpha}_h$ by (5) and (6).

This proposition shows under which condition there exist a unique Bayesian-Nash equilibrium and characterize it and a combination of Katz-Bonacich centralities.

3.2 Key-player policy

Let us now study the key-player policy in this model where the unknown parameter is α and there are two states of the world, α_l and α_h.[3] We assume that the planner has a prior (which is unknown to the agents) for the event $\{\alpha = \alpha_h\}$, which is given by:

$$\mathbb{P}(\{\alpha = \alpha_h\}) = p^A \in (0,1)$$

The prior of the planner may be different than the one shared by the agents, which is given by (4) because the planner may have superior information. The planner needs to solve the key player problem, which is the difference in aggregate activity according to her prior, i.e. $\max_{i \in \{1,\ldots,n\}} \Delta X_i$, where

$$\Delta X_i = \underbrace{\left[p^A \overline{x}^*(\mathbf{G}) + (1-p^A)\underline{x}^*(\mathbf{G})\right]}_{\text{Total activity before the removal of } i} - \underbrace{\left[p^A \overline{x}^*(\mathbf{G}^{-i}) + (1-p^A)\underline{x}^*(\mathbf{G}^{-i})\right]}_{\text{Total activity after the removal of } i}$$

where $\overline{x}^*(\mathbf{G}) = \sum_{j=1}^n \overline{x}_j^*$, $\overline{x}^*(\mathbf{G}^{-i}) = \sum_{j=1, j\neq i}^n \overline{x}_j^*$, $\underline{x}^*(\mathbf{G}) = \sum_{j=1}^n \underline{x}_j^*$ and $\underline{x}^*(\mathbf{G}^{-i}) = \sum_{j=1, j\neq i}^n \underline{x}_j^*$, and where $\overline{x}_j^*$ is the Bayesian-Nash equilibrium high action of agent j defined by (10) while $\underline{x}_j^*$ is the Bayesian-Nash equilibrium low action of agent j defined by (9). Indeed, if the planner believes that the state of the world is α_h, which occurs with probability p^A, then she believes that all agents will play the high actions while, if it is α_l, then she thinks that the low actions will be played.

Using the values defined in (10) and (9), we obtain:

$$\begin{aligned}
\Delta X_i &= \widehat{\alpha}\left[\sum_{j=1}^{n} b_j(\beta, \mathbf{G}) - \sum_{j=1}^{n-1} b_j\left(\beta, \mathbf{G}^{-i}\right)\right] \\
&\quad + \omega\left[\sum_{j=1}^{n} b_j\left((\gamma_h + \gamma_l - 1)\beta, \mathbf{G}\right) - \sum_{j=1}^{n-1} b_j\left((\gamma_h + \gamma_l - 1)\beta, \mathbf{G}^{-i}\right)\right]
\end{aligned}$$

where

$$\omega := (\widehat{\alpha}_h - \widehat{\alpha}_l)\left[p^A \frac{(1-\gamma_h)}{(2-\gamma_h-\gamma_l)} - (1-p^A)\frac{(1-\gamma_l)}{(2-\gamma_h-\gamma_l)}\right] \tag{12}$$

and $\widehat{\alpha}$ is defined in (11), γ_l and γ_h are given by (7) and (8), and $\widehat{\alpha}_l$ and $\widehat{\alpha}_h$ by (5) and (6). Using the definition of intercentrality given in (3), this is equivalent to

$$\Delta X_i = \widehat{\alpha}\, IC_i(\beta, \mathbf{G}) + \omega\, IC_i\left((\gamma_h + \gamma_l - 1)\beta, \mathbf{G}\right)$$

We have the following proposition:

[3] The generalization to a finite number of states of the world is relatively straightforward. Also, solving the key-player problem when the synergy parameter β is unknown (instead of α) leads to similar results. Both results are available upon request

Proposition 3 *Assume $\beta < 1/\lambda_{max}(\mathbf{G})$. Then, under incomplete information on α and with two states of the world, the key player is given by:*

$$\arg \max_{i \in \{1,\ldots,n\}} \{\Delta IC_i (\beta, \mathbf{G}, \mathbf{\Gamma}) \equiv \widehat{\alpha}\, IC_i (\beta, \mathbf{G}) + \omega\, IC_i ((\gamma_h + \gamma_l - 1) \beta, \mathbf{G})\}$$

If $\omega = 0$, then, when $\widehat{\alpha}_l - \widehat{\alpha}_h \to 0$, which means that both levels of αs (i.e. state of the world) are very similar, the optimal targeting is equivalent to the complete information case. If $\omega \neq 0$, then the optimal targeting may change. This is because the ranking derived from the intercentrality measure is not stable over β. Let us show with a simple example that, indeed, the key player may change between the complete information and incomplete information case.

3.3 Example

Consider the bridge network $\mathbf{g}^B$ in Figure 1 with eleven agents due to Ballester et al. (2006):

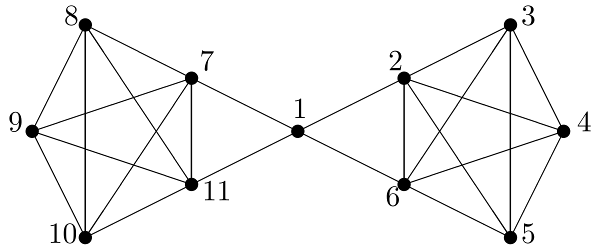

Figure 1: Bridge network

We distinguish three different types of equivalent actors in this network, which are of type 1 (player 1), type 2 (players 2, 6, 7, and 11) and type 3 (players 3, 4, 5, 8, 9, and 10). For the case of perfect information, Table 1 computes, for agents of types 1, 2 and 3 the value of the Katz-Bonacich centrality measures $b_i(\beta, \mathbf{G}^B)$ (which is equal to the effort x_i^* when $\alpha = 1$) and the intercentrality measures $IC_i(\beta, \mathbf{G}^B)$ for different values of β. In each column, a variable with a star identifies the highest value.

Type	$b_i(0.096, \mathbf{G}^B)$	$IC_i(0.096, \mathbf{G}^B)$	$b_i(0.2, \mathbf{G}^B)$	$IC_i(0.2, \mathbf{G}^B)$
1	1.679	2.726	7.143	35.714*
2	1.793*	3.016*	7.381*	28.601
3	1.646	2.570	6.191	21.949

Table 1: Katz-Bonacich versus intercentrality measures in a bridge network

Therefore, under complete information, the most active criminal (i.e. the one with the highest Katz-Bonacich centrality) is player 2 but the key player varies depending on the discount factor β. When β is small (equal to 0.096), then the key player is also the most active criminal. When β is larger (equal to 0.2), then the key player is *not* the most active criminal since it is player 1.

Let us now calculate the key player for the bridge network $\mathbf{g}^B$ with eleven agents displayed in Figure 1 when there is incomplete information on α. The key player is the agent i that maximizes

$$\Delta IC_i\left(\beta, \mathbf{G}^B, \mathbf{\Gamma}\right) \equiv \widehat{\alpha} IC_i\left(0.2; \mathbf{G}\right) + \omega IC_i\left(0.096; \mathbf{G}\right)$$

Using Table 1, we obtain:

Type	$b_i(0.096, \mathbf{G}^B)$	$IC_i(0.096, \mathbf{G}^B)$	$b_i(0.2, \mathbf{G}^B)$	$IC_i(0.2, \mathbf{G}^B)$	$\widehat{\alpha} IC_i(0.2) + \omega IC_i(0.096)$
1	1.679	2.726	7.143	35.714*	$35.714\,\widehat{\alpha} + 2.726\,\omega$
2	1.793*	3.016*	7.381*	28.601	$28.601\,\widehat{\alpha} + 3.016\,\omega$
3	1.646	2.570	6.191	21.949	$21.949\,\widehat{\alpha} + 2.57\,\omega$

Table 2: Intercentrality measures in a bridge network with imperfect information

Hence, the key player depends on the ratio $\omega/\widehat{\alpha}$, which is given by:

$$\frac{\omega}{\widehat{\alpha}} = \frac{\left(\widehat{\alpha}_h - \widehat{\alpha}_l\right)\left[p^A\left(1-\gamma_h\right) - \left(1-p^A\right)\left(1-\gamma_l\right)\right]}{\left(1-\gamma_h\right)\widehat{\alpha}_l + \left(1-\gamma_l\right)\widehat{\alpha}_h}$$

where γ_l and γ_h are given by (7) and (8), $\widehat{\alpha}_l$ and $\widehat{\alpha}_h$ by (5) and (6), and $\widehat{\alpha}$ by (11). This ratio captures all the (incomplete) information that the agents have: the priors p and p^A of the agents and the planner and the posteriors γ_l and γ_h of the agents. If the ratio $\omega/\widehat{\alpha}$ is small, type-1 agents are more likely to be the key players as in the perfect information case. However, if $\omega/\widehat{\alpha}$ is high, then type-2 agents will be the key players, which give the opposite prediction compared to the perfect information case. It readily verified that $\omega/\widehat{\alpha}$ increases with p^A, γ_h, and $\widehat{\alpha}_h$ and decreases with γ_h and $\widehat{\alpha}_l$. To understand this result, two effects need to be considered: (i) there is a *common effect* when $|\omega|$ is large (e.g. high *variance* of $\widehat{\alpha}$); (ii) there is an *idiosyncratic effect* when $\gamma_H + \gamma_L - 1 \ll 1$ and then the change in ranking is more likely to occur.

To illustrate this result, assume, for example, that $\alpha_l = 0.2$ and $\alpha_h = 0.8$, so that $\widehat{\alpha}_l = 0.326$, $\widehat{\alpha}_h = 0.737$ and $\widehat{\alpha} = 0.56$. Also assume that $p = 0.6$ and $q = 0.85$, so that $\gamma_l = 0.703$ and $\gamma_h = 0.776$. Finally, assume $p^A = 0.8 > p = 0.6$. Then using (12), we have: $\omega = 0.0945$, which means that $\omega/\widehat{\alpha} = 0.169$. It is easily verified that, when α is (partially) unknown, the key player is individual 1 since she is the one who has the highest $\widehat{\alpha} IC_i(0.2) + \omega IC_i(0.096)$, while, when there is perfect information on α, the key player is individual 2 if β is low enough (for example, if $\beta = 0.096$; see Table 2) and individual 1 if β is high enough (for example, if $\beta = 0.2$; see Table 2). If we now change the parameters of the model so that $\omega/\widehat{\alpha}$ becomes larger (by assuming higher values of p^A, γ_h, and $\widehat{\alpha}_h$ and lower values of γ_h and $\widehat{\alpha}_l$), then the key player in the imperfect information case becomes individual 2.

4 Conclusion

In this paper, we show that incomplete information can distort the network policy implications with respect to the complete information benchmark. Indeed, the targeting of individuals in a network (key-player policy) may be very different when information of the payoff structure is partially known. This implies that the planner may target the "wrong" individual.

We believe that, in many situations, information is incomplete and thus our model can shed light on these issues. For example, in criminal networks (Ballester et al., 2010; Calvó-Armengol and Zenou, 2004; Liu et al., 2013), delinquents do not always know the proceeds from crime. In R&D networks (Goyal and Moraga-Gonzalez, 2001; König et al., 2014), the marginal reduction in production costs due to R&D collaboration is partially known. In education (Calvó-Armengol et al., 2009), all students know each other connections in the classroom, i.e. the network structure, but they may not be completely aware of what are the benefits of studying. As a result, targeting the "right" key player important when information is incomplete. This is particularly relevant for crime (Liu et al., 2012; Lindquist and Zenou, 2014) but also for financial networks (Denbee et al., 2014), R&D networks (König et al., 2014) and training policies (Lindquist et al., 2015).

References

[1] Ballester C, Calvó-Armengol. A. and Y. Zenou (2006), "Who's who in networks: wanted the key player," *Econometrica* 74, 1403-1417.

[2] Ballester, C., Calvó-Armengol, A. and Y. Zenou (2010), "Delinquent networks," *Journal of the European Economic Association* 8, 34-61.

[3] Bonacich P. (1987), "Power and centrality: A family of measures," *American Journal of Sociology* 92, 1170-1182.

[4] Calvó-Armengol, A., E. Patacchini, and Y. Zenou (2009), "Peer effects and social networks in education," *Review of Economic Studies* 76, 1239-1267.

[5] De Martí, J. and Y. Zenou (2016), "Networks games under incomplete information," *Journal of Mathematical Economics*, forthcoming.

[6] Denbee, E., Julliard, C., Li, Y. and K. Yuan (2014), "Network risk and key players: A structural analysis of interbank liquidity," Unpublished manuscript, London School of Economics and Political Science.

[7] Goyal, S. and J.L. Moraga-Gonzalez (2001), "R&D networks," *RAND Journal of Economics* 32, 686-707.

[8] Ioannides, Y.M. (2012), *From Neighborhoods to Nations: The Economics of Social Interactions*, Princeton: Princeton University Press.

[9] Jackson M.O. (2008), *Social and Economic Networks*, Princeton, NJ: Princeton University Press.

[10] Jackson, M.O. and Y. Zenou (2015), "Games on networks", In: P. Young and S. Zamir (Eds.), *Handbook of Game Theory, Vol. 4*, Amsterdam: Elsevier Publisher, pp. 91-157.

[11] Katz, L. (1953), "A new status index derived from sociometric analysis," *Psychometrica* 18, 39-43.

[12] König, M.D., Liu, X. and Y. Zenou (2014), "R&D networks: Theory, empirics and policy implications," CEPR Discussion Paper No. 9872.

[13] Lindquist, M.J. and Y. Zenou (2014), "Key players in co-offending networks," CEPR Discussion Paper No. 9889.

[14] Lindquist, M.J., Sauermann, J. and Y. Zenou (2015), "Network effects on worker productivity," Unpublished manuscript, Stockholm University.

[15] Liu, X., Patacchini, E., Zenou, Y. and L-F. Lee (2012), "Criminal networks: Who is the key player?" CEPR Discussion Paper No. 8772.

[16] Zenou, Y. (2016), "Key players," In: Y. Bramoullé, B.W. Rogers and A. Galeotti (Eds.), *Oxford Handbook on the Economics of Networks*, Oxford: Oxford University Press, forthcoming.