Math II Final Exam

November 16, 2015

Jens Josephson

Instructions. This exam has 5 questions and a maximum score of 90 points, which together with the maximum score on the assignments give a total of 100 points. In order to pass, you need to obtain at least 50 points in total on the exam and the assignments. Motivate your answers clearly. If you think that a question is vaguely formulated, specify the conditions used for solving it. No calculators or other aids are allowed.

1. (25 points) Consider the function $f(x) = Ax$ for all $x \in \mathbb{R}^n$, where $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

   (a) Prove that $f(x)$ is a linear function.

   (b) Prove that $f(x)$ is an injective function.

   (c) Use Cramer's rule to find the solution to $f(x) = (2, 2, 2)$.

   (d) Compute the eigenvalues of $f(x)$.

   (e) What can be said about the angle between x and $f(x)$, for $x \neq \mathbf{0}$?

2. (10 points) Each of 2 cabinets identical in appearance has 2 drawers. Cabinet A contains a silver coin in each drawer, and cabinet B contains a silver coin in one of its drawers and a gold coin in the other. A cabinet is randomly selected, one of its drawers is opened, and a silver coin is found. What is the probability that there is a silver coin in the other drawer?

3. (15 points) Suppose X and Y are random variables with joint pdf $f(x, y) = 8xy$ for $0 \leq x \leq y \leq 1$.

   (a) Find the marginal pdfs of X and Y.

   (b) Are the two variables statistically independent?

   (c) Compute $E[X \mid Y = 1/2]$.

4. (20 points) Consider a random sample $X_1, ..., X_n$ of independent random variables, where each X_i has the pdf $f_{X_i}(x_i) = 2e^{-2x_i}$ for $x_i \geq 0$.

   (a) What is the name of the distribution of X_i?

   (b) Derive the moment-generating function of X_i and use it to compute the mean and variance of X_i.

   (c) Derive the pdf of $Q = X_1 + X_2$.

   (d) Define convergence in distribution and compute the limiting distribution of $\sqrt{n}\left(\ln\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \ln\left(E\left[X_i\right]\right)\right)$ as $n \to \infty$. (If you cannot solve (b), just assume the mean of X_i is μ and the variance σ^2).

5. (20 points) Consider a random sample $X_1, ..., X_n$ of independent random variables, where each X_i has the pmf $f_{X_i}(x_i) = p(1-p)^{x_i - 1}$ for $x_i = 1, 2, 3, ...$ and $p \in (0, 1)$.

   (a) What is the name of the distribution of X_i?

   (b) Compute $\Pr(X_1 \geq 4 \mid X_1 \geq 2))$.

   (c) Derive the maximum likelihood estimator: $\hat{p}$.

   (d) Is $\hat{p}$ consistent? Is it unbiased? (Hint: to answer the second question, you may want to consider the case when $n = 1$.)