# Mathematics III re-take exam. Stockholm Doctoral Program. August 21, 2015

Sebastian Koehne

**Instructions** Clearly state all steps towards the answer. Showing understanding of a working method is more important than getting all the algebra exactly correct. Calculators not capable of solving differential and/or difference equations are allowed. You may use a "cheat sheet" consisting of hand-written notes on one sheet of A4 paper (single or double-sided). No other aid is allowed.

There is no guarantee against the existence of typos or ambiguities in the questions. If you believe there is a typo or some missing information in a question, state your additional assumptions and interpretations clearly.

If you get stuck on a question, try to provide some arguments for how the problem should be solved and then go on to the other questions. It is also a good idea to read the whole exam before you start.

Your final grade will based on your performance in the exam (0-90 points) and in the homeworks (0-10 points). To pass the course you need a minimum of 50 points in total.

Good luck!

1. [20 points] Find the solution of the difference equation

$$x_{t+2} + 4x_{t+1} - 12x_t = 7t^2 + 2t - 6$$

that satisfies $x_0 = -3$ and $x_1 = 9$.

2. [15 points] Consider a standard autonomous first-order difference equation: $x_{t+1} = F(x_t)$. Suppose the equation has a stationary state $x^*$. Is it possible that $x^*$ is unstable and globally asymptotically stable at the same time? If yes, provide an example. If no, provide a proof.

3. [20 points] Consider the following dynamic optimization problem:

$$\max_{\{u_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} \ln u_t + \ln x_T \quad \text{subject to } x_{t+1} = x_t - u_t, \quad u_t \in (0, x_t),$$

with $x_0 > 0$ given.

   (a) Find the value functions $J_s(x)$ and the corresponding optimal controls $u_s^*(x)$ for $s = T, T-1$. [10 points]

   (b) Show that $J_{T-n}(x) = (n+1)\ln\frac{x}{n+1}$ for $n = 0, 1, 2, \ldots, T$ and find the optimal controls $u_{T-n}^*(x)$.[10 points]

4. [20 points] Consider the optimal control problem

$$\max_{u(t)\in[0,1]} \int_0^T (1 - u(t))(x(t))^2 dt$$
$$\text{s.t.} \quad \dot{x} = ux, \quad x(0) = 1, \quad x(T) \text{ free}$$

   (a) Write down the conditions of the maximum principle. [5 points]

   (b) Show that the adjoint function $p(t)$ must be strictly decreasing. [5 points]

   (c) Solve the problem when $T > \frac{1}{2}$. [10 points]

5. [15 points] Find the solution of the control problem

$$\max_{u(t)\in[-1,1]} \int_0^2 (x(t))^3 dt$$
$$\text{s.t.} \quad \dot{x} = u, \quad x(0) = 0, \quad x(2) = 0$$