Question 1 (25 p)

Consider an economy with an infinitely lived representative household where population grows at constant rate n and technology at constant rate g. The household head solves

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \widetilde{\beta}^t u\left(\frac{C_t}{L_t}\right), \tag{1}$$

where $\widetilde{\beta} = \beta(1+n) < 1$ is the effective discount factor, subject to the budget constraint

$$I_t + C_t = w_t L_t + R_t K_t - T_t, \tag{2}$$

where T_t denotes lump-sum taxes, and the evolution of the capital stock is given by

$$K_{t+1} = (1-\delta) K_t + I_t. \tag{3}$$

Firms in the economy are price-takers, operating on a perfectly competitive market. Production in the economy is determined by the production function

$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}, \tag{4}$$

where $0 < \alpha < 1$. The representative firm's problem is to choose K_t and L_t to maximize its real profit function

$$\Pi_t = K_t^{\alpha} (A_t L_t)^{1-\alpha} - w_t L_t - (1-\tau) R_t K_t, \tag{5}$$

where τ is a subsidy on the rental cost of capital.

a) Derive the household's and the firm's first order conditions.

b) Assume that the subsidy on the rental cost of capital is financed through a lump-sum tax on the household. Derive the model's equilibrium conditions under this assumption.

c) Assume that $u(C_t/L_t) = \log(C_t/L_t)$ and that the economy is in a steady state, where variables normalized by the number of effective workers, $A_t L_t$, are constant. Rewrite the equilibrium conditions in intensive form. Find the value of the subsidy that equalizes steady state capital per effective worker to the golden rule level of capital. How is the size of the required subsidy affected by β. Explain intuitively.

Question 2 (25 p)

Assume an OLG economy where individuals live at most two periods. All individuals live for the first period, but an individual only survives to the second period with probability $\gamma \in [0, 1]$. The individual does not know whether or not he is going to survive into the second period when deciding how much to save. The size of the generation born in period t is L_t. An individual born at time t has life-time utility

$$\ln C_{1t} + \gamma\beta \log C_{2t+1}, \tag{6}$$

where C_{1t} is the individual's consumption in the first period of his life and C_{2t+1} consumption in the second period (given that he is still alive in the second period). The individual is born without assets and does not wish to leave a bequest. Savings are invested in the aggregate capital stock, which yields a return of $r_t = R_{t+1} - \delta$.

a) Assume that the savings of the deceased individuals are wasted. The first and second period budget constraints for an individual born in period t are then given by

$$s_t + C_{1t} = w_t, \tag{7}$$
$$C_{2t+1} = (1 + r_t)\, s_t. \tag{8}$$

Derive an expression for the individual's optimal saving as a function of the wage today. Explain intuitively how the individual's savings is affected by the uncertainty of survival into the second period.

b) Assume instead that the savings of the deceased individuals (with interest) are distributed evenly among the individuals of the same generation that are still alive, through a transfer T_t per individual. Write down the individual's budget constraints under this assumption. Derive an expression for the individual's optimal saving as a function of the wage today. Compared to (a), will an individual save more or less for a given wage? Explain intuitively.
Hint: After deriving the first-order conditions, impose the condition that transfers must equal savings of the deceased with interest.

Question 3 (25 p)

Consider an economy where the household head solves

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t), \tag{9}$$

subject to the budget constraint

$$I_t + C_t = R_t K_t, \tag{10}$$

where the evolution of the capital stock is given by

$$K_{t+1} = (1-\delta) K_t + I_t. \tag{11}$$

Production in the economy is determined by the production function

$$Y_t = AK_t + K_t^{\alpha}, \tag{12}$$

where $A > 0$ is the (constant) level of technology, and $0 < \alpha < 1$. The representative firm's problem is to choose K_t to maximize its real profit function

$$\Pi_t = AK_t + K_t^{\alpha} - R_t K_t, \tag{13}$$

a) Show that the production function is such that the economy can grow forever. Explain intuitively!

b) Solve for the growth rate of the economy when it has reached a balanced growth path where output, consumption, and the capital stock grow at the same constant rate. Explain how the growth rate depends on the parameters of the model. What restrictions on the parameters are necessary for the growth rate to positive, yet not so fast that utility becomes unbounded?

c) Is the economy consistent with transitional dynamics, i.e., will a country that starts out with a lower level of output grow faster than a richer country during a transition period before the poor country has reached the balanced growth path? Explain intuitively.

Question 4 (25p)

Consider an economy where the social planner solves

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(C_t), \tag{14}$$

subject to the aggregate resource constraint

$$K_{t+1} + C_t = A_t K_t^{\alpha}, \tag{15}$$

where A_t is a technology shock that is either 2.5 or 3.5, each with probability 0.5.

a) Formulate the Bellman equation for the social planner's problem. Let A and K denote this period's technology and capital stock and A' and K' next period's technology and capital stock.

b) Assume that $\alpha = 1$ and $\beta = 0.5$ and that the capital stock only can take on the values $\{1, 2\}$. Guess that the initial value function $V^0(A, K)$ is

$$V^0(2.5, 1) = 0.6, \tag{16}$$
$$V^0(3.5, 1) = 1.4, \tag{17}$$
$$V^0(2.5, 2) = 1.8, \tag{18}$$
$$V^0(3.5, 2) = 2.2. \tag{19}$$

Use value function iteration to update the guess once, i.e., calculate $V^1(A, K)$. You can use the table on the next page to lookup approximate values of the *log* function.

c) Would you consider $V^1(A, K)$ having converged to the true value function? Motivate your answer!

d) If you, regardless your answer in (c), consider $V^1(A, K)$ to be the true value function, what is the policy function $K'(A, K)$? Assume that the planner chooses the lower level of capital if there is a "tie".

x	$\log(x)$
0	$-\infty$
0.5	-0.7
1	0
1.5	0.4
2	0.7
2.5	0.9
3	1.1
3.5	1.3
4	1.4
4.5	1.5
5	1.6
5.5	1.7
6	1.8