Mathematics III exam. Stockholm Doctoral Program. January 18, 2016

Sebastian Koehne

Instructions Clearly state all steps towards the answer. Showing understanding of a working method is more important than getting all the algebra exactly correct. Calculators not capable of solving differential and/or difference equations are allowed. You may use a "cheat sheet" consisting of hand-written notes on one sheet of A4 paper (single- or double-sided). The sheet will be collected after the exam. No other aid is allowed.

There is no guarantee against the existence of typos or ambiguities in the questions. If you believe there is a typo or some missing information in a question, state your additional assumptions and interpretations clearly.

If you get stuck on a question, try to provide some arguments for how the problem should be solved and then go on to the other questions. It is also a good idea to read the whole exam before you start.

Your final grade will based on your performance in the exam (0-90 points) and in the homeworks (0-10 points). To pass the course you need a minimum of 50 points in total.

Good luck!

1. [25 points] Consider the difference equation

$$x_{t+2} + x_{t+1} - 6x_t = 5^t + t \tag{1}$$

   (a) Find the general solution of the *homogeneous version* of equation (1). [5 points]

   (b) Use your result from (a) to find the general solution of (1). [15 points]

   (c) Is equation (1) globally asymptotically stable? Are there stationary states? If yes, are they globally asymptotically stable? [5 points]

2. [20 points] Consider the following stochastic dynamic optimization problem (as discussed in the lecture). A risk-averse individual wins a gamble with probability $p > 1/2$. She plays T rounds of stochastically independent gambles. In each round $t = 0, \dots, T-1$, she bets a fraction $u_t \in [0,1]$ of her current wealth x_t. If she wins the gamble in round t, she gains an amount equal to $u_t x_t$. If she loses, her wealth falls by $u_t x_t$. Her initial wealth $x_0 > 0$ is given, and the individual maximizes her (ex-ante) expected utility $\mathbb{E}[\ln x_T]$ of terminal wealth x_T. (That is, the individual derives no utility from wealth in periods $0, \dots, T-1$ but only from her wealth after the realization of the last gamble.)

   (a) By construction, because there is no gamble after period $T-1$, we have $J_T(x_T) = \ln x_T$. Find the value function $J_{T-1}(x_{T-1})$ and the corresponding optimal control $u^*_{T-1}(x_{T-1})$. [10 points]

   (b) Show by induction that there exists a constant B such that $J_{T-k}(x) = \ln x + kB$ for $k = 0, 1, \dots, T$. Moreover, determine the optimal controls $u^*_{T-k}(x)$ for $k = 0, 1, \dots, T-1$. [10 points]

3. [30 points] Let T, a, x_0 be positive constants. Consider the following optimal control problem:

$$\max_{u(t)\in[0,1]} \int_0^T (x(t) - u(t))dt$$
$$\text{s.t.} \quad \dot{x}(t) = au(t)e^{-2t} - x(t), \quad x(0) = x_0, \quad x(T) \text{ free}$$

(a) Write down the conditions of the maximum principle for this problem. Find an explicit expression for the co-state variable $p(t)$. [10 points]

(b) Solve the problem with $T = \ln 10$, $a = \frac{1}{2}$, $x_0 = 5$. [10 points]

(c) Solve the problem with $T = \ln 10$, $a = 5$, $x_0 = 5$. [10 points]

4. [15 points] **(Hard!)** Consider a first-order difference equation

$$x_{t+1} = f(x_t),$$

where $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Let $x^* = f(x^*)$ be a stationary state of the equation, with $f'(x^*) = 1$, $f''(x^*) > 0$. Show that x^* is unstable. *Hint:* Argue that, by the mean value theorem, there exist numbers c_t between x^* and x_t for all t such that

$$x_t - x^* = f'(c_{t-1})f'(c_{t-2}) \cdots f'(c_0)(x_0 - x^*).$$