Math II Final Exam
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Instructions. This exam has 5 questions and a maximum score of 90 points, which together with the maximum score on the assignments give a total of 100 points. In order to pass, you need to obtain at least 50 points in total on the exam and the assignments. Motivate your answers clearly. If you think that a question is vaguely formulated, specify the conditions used for solving it. No calculators or other aids are allowed.

1. (25 points) Consider the function $f(x) = Ax$, where $A = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$ and $x \in \mathbb{R}^2$.

   (a) Characterize $\mathcal{K}(f)$ and use the fundamental theorem of algebra to calculate $\dim[\mathcal{R}(f)]$.

   (b) Is f an injection? A surjection? Motivate.

   (c) Use Cramer's rule to calculate the solution to $f(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   (d) Find the eigenvalues and corresponding eigenvectors of length one of f.

   (e) Define positive definiteness and determine if the function has this property.

2. (10 points) Suppose the number of breakdowns of a new car during a year, X, is a random variable with probability mass function $f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, for $x = 0, 1, 2, \ldots$, where $\lambda = 1$. Suppose further that the car manufacturer promises to give the buyer of the car his money back if the car breaks down more than once during the first year after the purchase. The purchase price of the car is p.

   (a) What is the probability that the manufacturer has to give the buyer his money back?

   (b) Suppose the car breaks down exactly once during the first 6 months after its purchase. What are the expected value and variance of the reimbursement conditional upon this information?

3. (20 points) Suppose the time between breakdowns, Y, has the density function $f_Y(y) = e^{-y}$ for $y > 0$ and 0 otherwise.

   (a) What are the names of the distributions of X (in question 3.) and Y (in question 4.)?

   (b) Calculate the CDF of Y.

   (c) Derive the moment-generating function of Y and use it to compute $E[Y^3]$.

   (d) Derive the density function of $Q = \sqrt{2Y}$.

4. (15 points) Suppose S and T are random variables with joint density function $f_{ST}(s,t) = 5/4 - st$ for $s \in [0,1]$ and $t \in [0,1]$, and 0 otherwise.

   (a) Derive the marginal density function of S and determine if S and T are statistically independent.

   (b) Compute the value of the sum $Cov[S,T] + E[S]E[T]$.

   (c) Calculate $E[T \mid S = 1/2]$.

5. (20 points) Consider the sequence of independent and identically distributed random variables $V_1, V_2, ...$, where $E[V_i] = 1$ and $Var[V_i] = 1$. Let $\bar{V}_n = \frac{1}{n}\sum_{i=1}^{n} V_i$, for $n = 1, 2, ...$

   (a) Define convergence in probability and prove the Weak Law of Large Numbers in the setting of this question.

   (b) Define convergence in distribution and derive the limiting distribution of $\sqrt{n}\left(\bar{V}_n\right)^2$ as n tends to infinity.