Math II Retake Exam
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Instructions. This exam has 7 questions and a maximum score of 90 points, which together with the maximum score on the assignments give a total of 100 points. In order to pass, you need to obtain at least 50 points in total on the exam and the assignments. Motivate your answers clearly. If you think that a question is vaguely formulated, specify the conditions used for solving it. No calculators or other aids are allowed.

1. (15 points) Consider the function $f(x) = Ax$ for all $x \in \mathbb{R}^3$, where
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
   (a) Find the kernel $\mathcal{K}(f)$ and derive the dimension of the range $\mathcal{R}(f)$.
   (b) Calculate the eigenvalues and eigenvectors of A.
   (c) Define positive semi-definiteness and find out if f has this property or not.

2. (10 points) Show that the orthogonal complement $X^{\perp}$ to any subspace X of $\mathbb{R}^n$ is also a subspace of $\mathbb{R}^n$.

3. (10 points) A test indicates the presence of a particular disease 80% of the time when the disease is present and the presence of the disease 10% of the time when the disease is not present. If 10% of the population has the disease, calculate the probability that a person selected at random has the disease if the test indicates the presence of the disease.

4. (15 points) Suppose X and Y are random variables with joint pdf $f_{XY}(x, y) = 2$ for $0 \leq y \leq x \leq 1$ and 0 otherwise.
   (a) Find the marginal pdfs of X and Y.
   (b) Compute $E[Y \mid X = 1/2]$
   (c) Compute $Cov[X, Y]$.

5. (15 points) Consider a random sample $X_1, ..., X_n$ of independent random variables, where each X_i has the pdf $f_{X_i}(x_i) = 2x_i$ for $x_i \in [0, 1]$ and 0 otherwise.

   (a) Calculate the variance of X_i.

   (b) Derive the pdf of $Q = X_i^2$.

   (c) Define convergence in distribution and compute the limiting distribution of $\sqrt{n}\left(\ln\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \ln\left(E\left[X_i\right]\right)\right)$ as $n \to \infty$.

6. (15 points) Suppose a random variable X takes the value 1 with probability p and 0 with probability $1 - p$.

   (a) Write down the pmf of X.

   (b) Derive the moment-generating function of X and use it to compute the mean and variance of X.

   (c) Suppose we observe a random sample $X_1, ..., X_n$ of independent random variables, each distributed as X. Derive the maximum likelihood estimator of p: $\hat{p}$.

7. (10 points) The time (in minutes) a customer has to wait before being served in an ice-cream parlor is a random variable Y with pdf $g_Y(y) \geq 0$ for $y \geq 0$ and 0 otherwise. Assuming that the mean of Y is 2 minutes, prove that the probability that the customer is served within 8 minutes is at least 75%.