Mathematics III exam. Stockholm Doctoral Program. January 20, 2017

Sebastian Koehne, Alexandre Kohlhas

Instructions Clearly state all steps towards the answer. Showing understanding of a working method is more important than getting all the algebra exactly correct. Calculators not capable of solving differential and/or difference equations are allowed. You may use a "cheat sheet" consisting of hand-written notes on one sheet of A4 paper (single- or double-sided). The sheet will be collected after the exam. No other aid is allowed.

There is no guarantee against the existence of typos or ambiguities in the questions. If you believe there is a typo or some missing information in a question, state your additional assumptions and interpretations clearly.

If you get stuck on a question, try to provide some arguments for how the problem should be solved and then go on to the other questions. It is also a good idea to read the whole exam before you start.

Your final grade will based on your performance in the exam (0-90 points) and in the homeworks (0-10 points). To pass the course you need a minimum of 50 points in total.

Good luck!

1. [15 points] Consider a linear first-order difference equation:

$$x_{t+1} = a_t x_t + b_t \tag{1}$$

where $(a_t)_t$ and $(b_t)_t$ are given sequences of real numbers, and b_t is nonzero for any t.

Answer the following questions and carefully explain your answers.

(a) Under what conditions does equation (1) have stationary states? [5 points]

(b) If stationary states of (1) exist, under what conditions are they (locally or globally) asymptotically stable? [5 points]

(c) Can equation (1) have multiple stationary states? [5 points]

2. [15 points] Consider the difference equation

$$x_{t+2} - 6x_{t+1} + 25x_t = 1 \tag{2}$$

(a) Find the general solution of the associated *homogeneous* equation. [5 points]

(b) Use your result from (a) to find the general solution of equation (2). Are there stationary states? If stationary states exist, are they (locally or globally) asymptotically stable? [10 points]

3. [15 points] Consider the following homogeneous system of $n \geq 2$ first-order difference equations (written in matrix form):

$$X_{t+1} = AX_t$$

Suppose that the matrix A is diagonalizable. Show that the general solution to the above equation is given by

$$X_t = C_1\lambda_1^t v_1 + C_2\lambda_2^t v_2 + \cdots + C_n\lambda_n^t v_n,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $v_1, \ldots, v_n$ are the corresponding (linearly independent) eigenvectors, and $C_1, \ldots, C_n$ are arbitrary constants.

4. [20 points] Consider the following dynamic optimization problem,

$$\max_{u \in [-1,1]} \int_0^1 (2x - x^2) dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(1) = 0.$$

(a) Write down the conditions of the *maximum principle*. Are these conditions sufficient? [7 points]

(b) Show that $\lambda(t)$ is decreasing [7 points]
(*hint*: use the equation of motion)

(c) Suppose that $\lambda(t)$ has a unique crossing with the zero-line at $\hat{t} = 1/2$. Use this to derive $u^\star(t)$, $\lambda^\star(t)$ and $x^\star(t)$ [6 points]

5. [20 points] Consider the problem,

$$\max_{(u_t \in \mathbb{R})_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t(-u_t^2 - x_t^2), \quad \beta \in (0,1), \quad x_{t+1} = u_t + x_t, \quad x_0 \text{ given.}$$

(a) Guess that the value function is of the form $V(x) = ax^2 + b$. Determine a and b. [7 points]

(b) Consider the corresponding finite horizon problem (where the sum now is up to T). Assume that $V_t(x) = a_t x^2 + b_t$. Derive recursions for a_{t-1}, b_{t-1} and V_{t-1} as functions of a_t and b_t. [7 points]

(c) Determine an expression for $V_{T-2}(x)$. [6 points]

6. [5 points] Show that a contraction mapping $T : S \to S$ with modulus β cannot have multiple fixed points $V \in S$ (i.e. $TV = V$).