Mathematics III re-take exam. Stockholm Doctoral Program. August 2017

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Instructions Clearly state all steps towards the answer. Showing understanding of a working method is more important than getting all the algebra exactly correct. Calculators not capable of solving differential and/or difference equations are allowed. You may use a "cheat sheet" consisting of hand-written notes on one sheet of A4 paper (single- or double-sided). The sheet will be collected after the exam. No other aid is allowed.

There is no guarantee against the existence of typos or ambiguities in the questions. If you believe there is a typo or some missing information in a question, state your additional assumptions and interpretations clearly.

If you get stuck on a question, try to provide some arguments for how the problem should be solved and then go on to the other questions. It is also a good idea to read the whole exam before you start.

Your final grade will based on your performance in the exam (0-90 points) and in the homeworks (0-10 points). To pass the course you need a minimum of 50 points in total.

Good luck!

1. [20 points] Find the solution of the difference equation

$$x_{t+2} + 4x_{t+1} - 12x_t = 7t^2 + 2t - 6$$

that satisfies $x_0 = -3$ and $x_1 = 9$.

2. [25 points] Consider the following system of first-order difference equations (written in matrix form):

$$X_{t+1} = AX_t, \text{ where } A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

   (a) [10 points] Diagonalize the matrix A to find a simple formulation of the general solution.

   (b) [5 points] Is the system globally asymptotically stable? Explain.

   (c) [10 points] Use your result from (a) to find the general solution of the following nonhomogeneous equation:

$$X_{t+1} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} X_t + \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

3. [20 points] Consider the following dynamic optimization problem:

$$\max_{\{u_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} \ln u_t + \ln x_T \quad \text{subject to } x_{t+1} = x_t - u_t, \quad u_t \in (0, x_t),$$

with $x_0 > 0$ given.

   (a) Find the value functions $J_s(x)$ and the corresponding optimal controls $u_s^*(x)$ for $s = T, T-1$. [10 points]

   (b) Proceed by induction to find $J_s(x)$ and the optimal controls $u_s^*(x)$ for $s = 0, 1 \ldots, T-2$. [10 points]

4. [25 points] Solve the following optimal control problem:

$$\max_{u(t)\in[0,1]} \int_0^2 \left(x(t) - u(t)^2\right) dt$$

subject to

$$\dot{x}(t) = x(t) + u(t), \quad x(0) = 0, \quad x(2) \text{ free}, \quad u(t) \in [0, 1].$$