Mathematics III exam. Stockholm Doctoral Program. January 15, 2017

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Instructions: Clearly state all steps towards the answer. Showing understanding of a working method is more important than getting all the algebra exactly correct. Calculators not capable of solving differential and/or difference equations are allowed. You may use a "cheat sheet" consisting of hand-written notes on one sheet of A4 paper (single- or double-sided). The sheet will be collected after the exam. No other aid is allowed.

There is no guarantee against the existence of typos or ambiguities in the questions. If you believe there is a typo or some missing information in a question, state your additional assumptions and interpretations clearly.

If you get stuck on a question, try to provide some arguments for how the problem should be solved and then go on to the other questions. It is also a good idea to read the whole exam before you start.

Your final grade will based on your performance in the exam (0-90 points) and in the homeworks (0-10 points). To pass the course you need a minimum of 50 points in total.

Good luck!

1. [15 points] Consider the differential equation $\dot{x} + 2x = 1$

   (a) Find the general solution in two ways: (i) separating variables, (ii) by use of an integrating factor. [10 points]

   (b) The equation is autonomous. Sketch its phase diagram and indicate stationary points. Are they stable? [5 points]

2. [15 points] Consider second-order differential equations.

   (a) Find the general solution to $\ddot{x} + 2\dot{x} + 2x = 0$ [5 points]

   (b) A second-order linear differential equation is asymptotically stable iff the roots of the associated characteristic equation have negative real parts. Consider the equation $\ddot{x} + 2\dot{x} + cx = 0$, c constant.

      i. Indicate which values of c correspond to two real roots, repeated real roots and complex roots of the associated characteristic equation. [2 points]

      ii. For the case of two real roots, indicate when they are both negative, both positive or have different signs. [2 points]

      iii. Summarize this information by drawing an axis and marking on it different intervals corresponding to the possibilities for the roots. [3 points]

      iv. Indicate for which intervals the differential equation is stable. [3 points]

3. [15 points] Consider the following first-order difference equations:

   a) Solve $x_{t+1} = 0.5x_t + 3$. Find its steady state and decide whether it is stable or not. Describe (in words or graph) the long-run behavior of the solution. [5 points]

   b) Consider the first-order difference equation

   $$x_{t+1} = a_t x_t, \quad t = 0, 1, 2, \ldots,$$

   where $(a_t)_t$ is a sequence of real numbers. Show that the general solution is given by

   $$x_t = \left(\prod_{s=0}^{t-1} a_s \right) x_0, \quad t = 0, 1, 2, \ldots.$$

Recall: We define $\prod_{s=k}^{l} a_s = a_k a_{k+1} \cdots a_l$ for $l = k, k+1, k+2, \ldots$, and $\prod_{s=k}^{l} a_s = 1$ for $l < k$. [5 points]

c) Let $\alpha \in$. Find the general solution to

$$x_{t+1} = \frac{3}{2t+3} x_t, \quad t = 0, 1, 2, \ldots,$$

and examine its long-run behavior ($t \to \infty$). [5 points]

4. [20 points] Consider the dynamic optimization problem

$$\max_{u \in \mathbb{R}} \int_0^1 (x+u)dt, \quad \dot{x} = 1 - \frac{1}{2}u^2, \quad x(0) = x_0, \quad x(1) \geq 0.$$

(a) Write down the conditions of the *maximum principle.* [6 points]

(b) Solve the problem. [8 points]

(c) Show that $\frac{\partial V(x_0,T)}{\partial x_0} = \lambda(0)$. [6 points]

5. [20 points] Consider the dynamic optimization problem

$$\max \sum_{t=0}^{T-1} 2\sqrt{u_t x_t} + \sqrt{x_T}, \quad u_t \in [0,1], \quad x_{t+1} = (1-u_t)x_t, \quad x_0 \text{ given.}$$

(a) Find $V_{T-1}(x)$, $V_{T-2}(x)$, $u^\star_{T-1}$, and $u^\star_{T-2}$. [7 points]

(b) Show that V_t can be written in the form $V_t(x) = k_t\sqrt{x}$, and find a difference equation for k_t. [7 points]

(c) Consider the infinite horizon extension (where the sum $\to \infty$). Show that x_t satisfies a second-order difference equation. [6 points] (*hint*: use the euler equation.)

6. [5 points] Consider the *bellman equation*

$$V(x) = \max_{x' \in \Gamma(x)} \left[F(x,x') + \beta V(x')\right], \quad \beta \in (0,1).$$

Suppose the problem is well-defined; a solution exists and F is continuous and bounded. Suppose, furthermore, that F is strictly increasing in x and that the feasibility correspondence Γ is monotone such that if $x \leq \hat{x}$, then $\Gamma(x) \subseteq \Gamma(\hat{x})$. Show that $V(x)$ is strictly increasing.