Macro I, retake exam

August 15, 2018

Directions

The exam yields a total of 100 points. Provide **brief and concise** answers. Keep auxiliary computations **separate** from your main results. Write **legibly** (**pen** $\succ$ **pencil**). Thank you and good luck.

Problem 1: Solow model (15 points)

Consider the following Solow setup: The law of motion of the capital stock is

$$K_{t+1} = K_t(1-\delta) + qI_t \tag{1}$$

The saving behavior is

$$I_t = sF\left[K_t, A_tL_t\right] \tag{2}$$

and the resource constraint is

$$F\left[K_t, A_tL_t\right] = C_t + I_t. \tag{3}$$

$F\left[K_t, A_tL_t\right]$ is a neoclassical production function that fulfills the standard assumptions. We have $A_t = \gamma^t$ and $L_t = \eta^t$. $q > 0, 1 > s > 0$, $\gamma > 1$ and $\eta > 1$ are some parameters.

(a) **(5 points)** Let us focus on the steady state where $k_t \equiv \frac{K_t}{A_tL_t}$ is constant. Calculate the capital-output ratio, $\frac{K_t}{F[K_t, A_tL_t]}$, in this stationary point. Is this capital-output ratio increasing or decreasing in q? Give an intuition.

(b) **(3 points)** The production function takes the following functional form:

$$F\left[K_t, A_tL_t\right] = \left[\alpha K_t^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)\left(A_tL_t\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}. \tag{4}$$

Calculate the golden rule capital stock per efficiency unit, $k^{\star}_{gold}$, for this economy.

(c) **(7 points)** Suppose instead that the production function takes the following functional form

$$F\left[K_t, A_tL_t\right] = K_t^{\alpha}\left(A_tL_t\right)^{1-\alpha}. \tag{5}$$

Calculate again the golden rule capital stock per efficiency unit, $k^{\star}_{gold}$, for this case. Is $k^{\star}_{gold}$ increasing or decreasing in q? Give an intuition. What saving rate, s, would lead to the golden rule capital stock?

(d) **(3 points)** Suppose $q = 2$, $s = 0.2$ $\delta = 0.04$, and $\eta\gamma = 1.04$. Calculate the steady state capital output ratio (from (a)) under this parametrization.

Problem 2: A closed form solution of the neoclassical growth model (22 points)

Consider the following neoclassical growth model: The preferences are

$$\mathcal{U}_0 = \sum_{t=0}^{\infty} (\beta\eta)^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \tag{6}$$

and the resource constraint is given by

$$\gamma\eta k_{t+1} = \left[\alpha k_t^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)\right]^{\frac{\epsilon}{\epsilon-1}} - \frac{c_t}{\gamma^t}. \tag{7}$$

k_0 is exogenously given. The transversality condition can be expressed as

$$\lim_{T\to\infty} \left[k_{T+1} (\beta\eta)^T \gamma^T c_T^{-\sigma}\right] = 0. \tag{8}$$

We have $\beta\eta\gamma^{1-\sigma} < 1$ and $\eta \geq 1$. Furthermore, we assume that the following knife-edge restriction is fulfilled

$$\epsilon\sigma = 1. \tag{9}$$

Since this restriction holds true, ϵ can be replaced everywhere by $1/\sigma$.

(a) **(7 points)** Write down the planner's problem and solve for the first-order conditions. Derive a system of first-order difference equations that characterizes the Pareto optimal allocation.

(b) **(5 points)** Is there a steady-state capital stock $k^\star$? If yes solve for it.

(c) **(5 points)** Show that for any k_t the Pareto optimal consumption is given by $c_t = z\gamma^t \left[\alpha k_t^{1-\sigma} + (1-\alpha)\right]^{\frac{1}{1-\sigma}}$ where z is some constant. Derive the precise value for z. Show that such a sequence is consistent with the transversality condition.

(d) **(5 points)** Calculate the approximate speed of convergence around the steady state μ where $k_{t+1} - k^\star \approx \mu[k_t - k^\star]$. Is the steady state locally saddle path stable?

Problem 3: Factor shares (10 points)

Describe (or draw) the dynamics of the labor income share in a developed country like the U.S.? What is roughly the level? How did it change over the last 80 years? How do the dynamics look for the last two decades? Elaborate a little bit on the following questions: Are there any challenges in terms of measuring the labor income share? How is it measured in practice? Will a decreasing capital-output ratio automatically lead to an increasing labor income share? [You might find it useful to argue with a CES poduction function] What are potential sources of a changing labor income share?

Exercise 4: A Competitive Equilibrium (20 points)

A pure endowment economy consists of two consumers $i = \{1, 2\}$. Consumers order consumption streams according to,

$$U^i = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \log\left[c_t^i\left(s^t\right)\right] \pi_t\left(s^t\right),$$

where $c_t^i \geq 0$, $\beta \in (0, 1)$ is a common discount factor, and $\pi_t\left(s^t\right)$ is the probability of the history $s^t = \left(s_0, s_1, ..., s_t\right)$. The consumption good is tradable but non-storable. The consumer of type 1 is endowed with the consumption sequence

$$y_t^1 = s_t \quad \forall t > 0,$$

where s_t is a random variable governed by a two-state Markov Chain with values $s_t = \bar{s}_1 = 1$ or $s_t = \bar{s}_2 = 0$. The consumer of type 2 is, by contrast, endowed with the consumption sequence

$$y_t^2 = 2 \quad \forall t > 0.$$

The aggregate endowment is denoted by $Y_t(s_t) = y_t^1 + y_t^2$.

(a) [**4 points**] Give an expression for $\pi_t\left(s^t\right)$.

(b) [**4 points**] Compute the competitive equilibrium price system with time-zero trading.

(c) [**4 points**] Consider the Planner's problem in which $\theta \in (0, 1)$ denotes the relative weight on consumer 1. Solve the Planner's problem and compare the solution to (b).

(d) [**4 points**] Consider instead sequential trading in one-period ahead Arrow securities. How many of such prices are there? Compute them when $\beta = 0.90$ and $\pi(s_j \mid s_i) = p_{ij}$, where

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}.$$

(e) [**4 points**] A new asset is introduced to the one-period Arrow economy. One of the households decides to market a two-period ahead riskless claim to 0.5 units of consumption. Compute its equilibrium price when $s_t = 1$. What about $s_t = 0$?

Exercise 5: Why Are Real Rates So Low? (20 points)

Assume a production function $Y_t = F(K_t, N_t)$, and the simplest two-period-lived overlapping generations model. The population is constant and normalized to one. Individuals supply inelastically one unit of labor in the first period of their lives.

Denote by U_t the life-time utility of the t'th generation,

$$U_t = \frac{\left(c_t^t\right)^{1-\theta}}{1-\theta} + \beta \frac{\left(c_{t+1}^t\right)^{1-\theta}}{1-\theta}, \quad \theta > 0. \tag{10}$$

Individual budget constraints for t and $t+1$ are

$$c_t^t + s_t^t = w_t \tag{11}$$

$$c_{t+1}^t = (1 + r_{t+1})\, s_t^t. \tag{12}$$

Suppose the relative price of one unit of investment (and hence capital) in terms of consumption is equal to p_t, and that the depreciation rate of capital is $\delta \in (0, 1)$.

(a) [**5 points**] Consider the relationship $r_t = \frac{1}{p_t}\frac{\partial Y_t}{\partial K_t} - \delta$. What is the intuition behind this relationship? State the capital markets clearing condition.[1]

(b) [**5 points**] Solve the t'th generation's problem. Show that the *savings rate* s_t

$$s_t = \frac{w_t - c_t^t}{w_t} = \frac{\beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}-1}}{1 + \beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}-1}} \tag{13}$$

State the condition for when $\frac{\partial s_t}{\partial r_{t+1}} > 0$.

(c) [**5 points**] Suppose that $F(\cdot)$ is a CES production function; that is, that

$$Y_t = \left[\alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha) N_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1 \tag{14}$$

Derive the capital-to-output ratio, K_t/Y_t, using the expression for r_t from (a). Use Euler's Theorem[2] to derive a relationship between W_t/Y_t and K_t/Y_t.

(d) [**5 points**] Use the results from (c) to derive a relationship between s_t and p_t. Derive conditions for when $\frac{ds_t}{dp_t} < 0$ and $\frac{dr_{t+1}}{ds_t} < 0$. Use these expressions to discuss how a fall in the relative price of investment goods could lead to low real rates (secular stagnation).

[1] Hint: the total value of capital has to equal what?

[2] Here, $N_t \frac{\partial Y_t}{\partial N_t} + K_t \frac{\partial Y_t}{\partial K_t} = Y_t$

Exercise 6: Investment in Africa (10 points)

A recent article in the Financial Times states that "*... the democratization of medicine has increased survival rates in Sub-Saharan Africa, pushing up investment returns.*"

Consider the simplest two-period-lived overlapping generations model with log-utility and Cobb-Douglas production. The number of young born at time t evolves according to the difference equation $N_t = (1+n)N_{t-1}$, where $n > 0$.

How does an increase in the population growth rate n affect individual and aggregate saving rates? How does an increase in n affect the return on investment in capital? Discuss these questions in light of the above comments about investment returns in Africa.