Mathematics III retake exam. Stockholm Doctoral Program. August 17, 2018

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Instructions: Clearly state all steps towards the answer. Showing understanding of a working method is more important than getting all the algebra exactly correct. Calculators not capable of solving differential and/or difference equations are allowed. You may use a "cheat sheet" consisting of hand-written notes on one sheet of A4 paper (single- or double-sided). The sheet will be collected after the exam. No other aid is allowed.

There is no guarantee against the existence of typos or ambiguities in the questions. If you believe there is a typo or some missing information in a question, state your additional assumptions and interpretations clearly.

If you get stuck on a question, try to provide some arguments for how the problem should be solved and then go on to the other questions. It is also a good idea to read the whole exam before you start.

Your final grade will based on your performance in the exam (0-90 points) and in the homeworks (0-10 points). To pass the course you need a minimum of 50 points in total.

Good luck!

1. [15 points] Let $a, b > 0$. Consider the differential equation $\dot{x} + ax = b$

   (a) Find the general solution in two ways: (i) separating variables, (ii) by use of an integrating factor.

   (b) The equation is autonomous. Sketch its phase diagram and indicate stationary points for some $a, b > 0$. Are they stable?

2. [15 points] Consider the logistic equation

$$\dot{y} \;=\; ay - by^2, \quad a, b > 0$$

   (a) What type of process is modelled by this equation?

   (b) Find the general solution to this equation.

   (c) Draw its phase diagram. Indicate the steady states and their stability.

3. [15 points] Consider the difference equation

$$x_{t+2} - 5x_{t+1} + 6x_t = 4^t + t^2 + 3 \tag{1}$$

   (a) Find the general solution of the *homogeneous version* of equation (1).

   (b) Use your result from (a) to find the general solution of (1).

   (c) Is equation (1) globally asymptotically stable?

4. [20 points] Consider the following dynamic optimization problem

$$\max_{u_t} \sum_{t=0}^{T} \ln u_t + x_t \quad \text{when} \quad x_{t+1} = x_t - u_t, \quad u_t \in (0, 1],$$

   with $x_0 > 0$ given.

   (a) Find $V_T(x), u_T^\star(x), V_{T-1}(x), u_{T-1}^\star(x), V_{T-2}(x)$, and $u_{T-2}^\star(x)$.[10 points]

   (b) Show that $V_{T-t}(x) = (1 + t)x - t - \ln\left(\Pi_{i=1}^{t} i\right)$.[10 points]

5. [20 points] Solve the following optimal control problem

$$V = \max_{u(t) \in \mathbb{R}} \int_0^T - \left(1 + u^2\right)^{1/2} dt$$

subject to

$$\dot{x}(t) = u(t), \quad x(0) = A, \quad x(T) \text{ free.}$$

Show that $\frac{\partial V}{\partial x_0} = \lambda(0)$.

6. [5 points] Let (S, ρ) be a complete metric space and let $T : S \to S$ be a contraction mapping with fixed point $v \in S$. If S' is a closed subset of S and $T(S') \subset S'$, then show that $v \in S'$.